# Chapter 1. Vectors, Matrices, and Linear Spaces

## 1.6. Homogeneous Systems, Subspaces and Bases

**Note.** In this section we explore the structure of the solution set of systems of equations. We also explore bases of subspaces of $\mathbb{R}^n$. First we define "homogeneous system." Later, we'll see that the general solution to consistent system $A\vec{x} = \vec{b}$ is related to the general solution to the homogeneous system $A\vec{x} = \vec{0}$.

**Definition.** A linear system $A\vec{x} = \vec{b}$ is *homogeneous* if $\vec{b} = \vec{0}$. The zero vector $\vec{x} = \vec{0}$ is a *trivial solution* to the homogeneous system $A\vec{x} = \vec{0}$. Nonzero solutions to $A\vec{x} = \vec{0}$ are called nontrivial solutions.

**Theorem 1.13. Structure of the Solution Set of $A\vec{x} = \vec{0}$.**
Let $A\vec{x} = \vec{0}$ be a homogeneous linear system. If $\vec{h_1}, \vec{h_2}, \ldots, \vec{h_n}$ are solutions, then any linear combination

$$r_1\vec{h_1} + r_2\vec{h_2} + \cdots + r_n\vec{h_n}$$

is also a solution.

**Proof.** Since $\vec{h_1}, \vec{h_2}, \ldots, \vec{h_n}$ are solutions, $A\vec{h_1} = A\vec{h_2} = \cdots = A\vec{h_n} = \vec{0}$ and so

$$A(r_1\vec{h_1} + r_2\vec{h_2} + \cdots + r_n\vec{h_n}) = r_1A\vec{h_1} + r_2A\vec{h_2} + \cdots + r_nA\vec{h_n} = \vec{0} + \vec{0} + \cdots + \vec{0} = \vec{0}.$$

Therefore the linear combination is also a solution. ∎

**Note.** We now turn our attention to subspaces of $\mathbb{R}^n$.

**Definition 1.16.** A subset $W$ of $\mathbb{R}^n$ is *closed under vector addition* if for all $\vec{u}, \vec{v} \in W$, we have $\vec{u} + \vec{v} \in W$. If $r\vec{v} \in W$ for all $\vec{v} \in W$ and for all $r \in \mathbb{R}$, then $W$ is *closed under scalar multiplication*. A nonempty subset $W$ of $\mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$ if it is both closed under vector addition and scalar multiplication.

**Example.** Page 99 Number 8.

**Theorem 1.14. Subspace Property of a Span**
Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$. Then $W$ is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are said to *span* or *generate* the subspace.)

**Example.** Page 100 Number 18.

**Definition.** Given an $m \times n$ matrix $A$, the span of the row vectors of $A$ is the *row space* of $A$, the span of the column vectors of $A$ is the *column space* of $A$ and the solution set to the system $A\vec{x} = \vec{0}$ is the *nullspace* of $A$.

**Definition 1.17.** Let $W$ be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ of $W$ is a *basis* for $W$ if every vector in $W$ can be expressed uniquely as a linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$.

**Theorem 1.15. Unique Linear Combinations.**

The set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ if and only if

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$$

implies

$$r_1 = r_2 = \cdots = r_k = 0.$$

**Example.** Page 100 Number 22(a).

**Note.** We now give additional conditions equivalent to the invertibility of matrix $A$ (thus extending Theorem 1.12, "Condition for $A^{-1}$ to Exist").

**Theorem 1.16.** Let $A$ be an $n \times n$ matrix. The following are equivalent:

**(1)** $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in \mathbb{R}^n$,

**(2)** $A$ is row equivalent to $\mathcal{I}$,

**(3)** $A$ is invertible, and

**(4)** the column vectors of $A$ form a basis for $\mathbb{R}^n$.

**Example.** Page 100 Number 22(b).

**Note.** We now explore some conditions under which the system $A\vec{x} = \vec{b}$ has a unique or multiple solutions where $A$ is possibly not square.

**Theorem 1.17.** Let $A$ be an $m \times n$ matrix. The following are equivalent:

**(1)** each consistent system $A\vec{x} = \vec{b}$ has a unique solution,

**(2)** the reduced row-echelon form of $A$ consists of the $n \times n$ identity matrix followed by $m - n$ rows of zeros, and

**(3)** the column vectors of $A$ form a basis for the column space of $A$.

**Corollary 1. Fewer Equations then Unknowns**

If a linear system $A\vec{x} = \vec{b}$ is consistent and has fewer equations than unknowns, then it has an infinite number of solutions. Such a system is called *underdetermined.*

**Corollary 2. The Homogeneous Case**

**(1)** A homogeneous linear system $A\vec{x} = \vec{0}$ having fewer equations than unknowns has a nontrivial solution (i.e. a solution other than $\vec{x} = \vec{0}$),

**(2)** A square homogeneous system $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if $A$ <u>is not</u> row equivalent to the identity matrix.

**Example.** Page 97 Example 6. A basis of $\mathbb{R}^n$ cannot contain more than $n$ vectors.

**Note.** We now relate the general solution of the homogeneous system $A\vec{x} = \vec{0}$ to the general solution of the system $A\vec{x} = \vec{b}$.

**Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.**

Let $A\vec{x} = \vec{b}$ be a linear system. If $\vec{p}$ is any particular solution of $A\vec{x} = \vec{b}$ and $\vec{h}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. In fact, every solution of $A\vec{x} = \vec{b}$ has the form $\vec{p} + \vec{h}$ and the general solution is $\vec{x} = \vec{p} + \vec{h}$ where $A\vec{h} = \vec{0}$ (that is, $\vec{h}$ is an arbitrary element of the nullspace of $A$).

**Examples.** Page 101 Numbers 36, Page 101 Number 43, Page 101 Number 47.

*Last modified: 2/18/2020*