Chapter 1. Vectors, Matrices, and Linear Spaces

1.7. Applications to Population Distributions

Note. In this section we break a population into "states" and gives a numerical description (a "transition matrix") of how the population changes states over discrete time increments. In particular, we are interested in the long term behavior of the population and we'll use this interest to motivate future topics (namely, eigenvalues, eigenvectors, and diagonalization in Chapter 5).

Note. We consider a population of individuals and partition the population into "states." For example, we might have a population of people partitioned into three economic states: poor, middle income, and rich (to use Fraleigh and Beauregard's terminology).

Definition. If a population is partitioned into states then we represent the proportion of the population in the states with a *population distribution vector* $\vec{p}$ where p_i is the proportion of the population in state i for $i = 1, 2, \ldots, n$.

Note. We now describe how the population changes with time. We do so using a matrix and a numerical description of how the states change. For example, if half of the individuals in state j move to state i over a time increment then we set $t_{ij} = 1/2$. Since there are n states this requires n^2 such numerical values.

Definition. If a population is partitioned into n states and the proportion t_{ij} of the population in state j moves to state i over a (discrete) time increment then we create the $n \times n$ *transition matrix* $T = [t_{ij}]$.

Page 103 Example 1. Let the population of a country be classified according to income as:

- State 1: poor,
- State 2: middle income,
- State 3: rich.

Suppose that over each 20-year period (about one generation) we have the following data for people and their offspring:

- Of the poor people, 19% become middle income and 1% rich.
- Of the middle class people, 15% become poor and 10% rich.
- Of the rich people, 5% become poor and 30% middle income.

Then we translate the percentages into propositions to get

$$\begin{array}{lll} t_{11} = 0.80 & t_{12} = 0.15 & t_{12} = 0.05 \\ t_{21} = 0.19 & t_{22} = 0.75 & t_{23} = 0.30 \\ t_{31} = 0.01 & t_{32} = 0.10 & t_{33} = 0.65 \end{array}.$$

(Notice that t_{ij} describes movement from state j to state i.) So the transition matrix is $T = \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.19 & 0.75 & 0.30 \\ 0.01 & 0.10 & 0.65 \end{bmatrix}$.

Note. Since the individuals in state j must move to one of the n states (including staying in state j) over each time increment then $\sum_{i=1}^{n} t_{ij} = 1$ for each $j = 1, 2, \ldots, n$. That is, the sum of each column of a transition matrix must be 1.

Note. If $\vec{p}$ is a population distribution (column) vector and T is a transition matrix, then $T\vec{p}$ is the population distribution vector after one time increment. In general, $\underbrace{TT \cdots T}_{k \text{ times}}\vec{p} = T^k\vec{p}$ is the population distribution vector after k time increments.

Note. Transition matrices and population distribution vectors are often used in the study of the distribution of genes in a population. We now cover Exercise 34–38 which concern such applications.

Note. On pages 113-114 in Numbers 34–38 we consider a simple Mendelian genetics model in which a particular trait can be affected by two genes, say G and g. Since our chromosomes come in pairs, such genes would appear in pairs giving three possible genotypes, GG (homozygous dominant), Gg (heterozygous or hybrid), and gg (homozygous recessive). An individual receives one gene from each parent at random. We use the population distribution vector $[p_1, p_2, \ldots, p_n]^T$ where p_1 is the proportion of genotypes GG, p_2 is the proportion of Gg, and p_3 is the proportion of type gg. Assume that a member of the population always reproduces with a heterozygous individual.

Example. Page 113 Number 34, Page 113 Number 35, Page 113 Number 36, Page 114 Number 37, Page 114 Number 38.

Note. The genetic model above is an example of a "Markov chain." this is a process described by a transition matrix from one time step to the next. There are certainly processes for which one would expect the translation matrix to change with time. For example, the matrix in Example 1 concerning poor/middle class/rich would expect to change as national or global economic circumstances change.

Definition. A *Markov chain* is an $n \times n$ transition matrix T, in which all entries of T are nonnegative and the sum of the column entries are 1 in each column, along with an initial population distribution vector $\vec{p}$. T is taken to be constant (and not changing with time increments) such that the series of population distribution vectors $\vec{p}$, $T\vec{p}$, $T^2\vec{p}$, $\ldots$ is produced.

Note. We now consider special Markov transition matrices.

Definition 1.19. Regular Transition Matrix, Regular Chain.
A transition matrix T is *regular* if T^m has no zero entries fro some integer m. A Markov chain having a regular transition matrix is a *regular chain*.

Example 1.7.A. In the genetic model of Exercises 34–39, the transition matrix is

$$T = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

and, as seen in Exercise 35,

$$T^2 = \begin{bmatrix} 3/8 & 1/4 & 1/8 \\ 1/2 & 1/2 & 1/2 \\ 1/8 & 1/4 & 3/8 \end{bmatrix}$$

so with $m = 2$ we see that T is regular.

Definition. In a Markov chain with transition matrix T, if there is a population distribution vector $\vec{s}$ such that $T\vec{s} = \vec{s}$, then $\vec{s}$ is a *steady state distribution vector.*

Note. The following result, the proof of which is “beyond the scope of this book” (page 108) shows that a regular Markov chain has a steady-state distribution vector.

Theorem 1.19. Achievement of steady state.

Let T be a regular transition vector. There exists a unique column vector $\vec{s}$ with strictly positive entries whose sum is 1 such that the following hold:

(1) As $m \to \infty$, all columns of $T^m \to \vec{s}$.

(2) $T\vec{s} = \vec{s}$ and $\vec{s}$ is the unique column vector with this property and whose column components add up to one.

Note 1.7.A. We wish to find the unique steady state distribution vector for a regular Markov chain. Since we need $T\vec{s} = \vec{s}$ for regular transition matrix T, we can consider the equivalent condition $T\vec{s} - \vec{s} = \vec{0}$ or $(T - \mathcal{I})\vec{s} = \vec{0}$. This is a homogeneous system of linear equations (with vector $\vec{s}$ of unknowns). By Theorem 1.19, "Achievement of Steady State," we know that there is a nonzero solution, namely a solution $\vec{s}$ the sum of whose components are 1. So our knowledge of homogeneous systems of equations allow us to find the steady state solution $\vec{s}$.

Examples. Page 112 Number 24, Page 114 Number 39.

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