Chapter 2. Dimension, Rank, and Linear Transformations

2.1. Independence and Dimension

Note. In Section 1.6 we defined a basis for a subspace of $\mathbb{R}^n$ (see Definition 1.17). In this section we explore some properties of such a basis and prove that every subspace of $\mathbb{R}^n$ actually does have a basis (see Theorem 2.3(1)). We are particularly interested in the size of a basis.

Definition 2.1. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a set of vectors in $\mathbb{R}^n$. A *dependence relation* in this set is an equation of the form

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$$

with at least one $r_j \neq 0$. If such a dependence relation exists, then $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a *linearly dependent* set. A set of vectors which is not linearly dependent is *linearly independent*.

Note 2.1.A. Notice that Definition 2.1 gives that for a linearly independent set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$, the equation $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$ implies $r_1 = r_2 = \cdots = r_k = 0$. In Exercise 29 it is shown that a set of two vectors $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

Theorem 2.1. Alternative Characterization of Basis

Let W be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ of W is a basis for W if and only if

(1) $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and

(2) the vector $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent.

Theorem 2.1.A. Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$.

Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \mathbb{R}^n$ and let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$. Form the matrix A whose jth column vector is $\vec{w}_j$. If we row-reduce A to row-echelon form H, then the set of all $\vec{w}_j$ such that the jth column of H contains a pivot, is a basis for W.

Examples. Page 134 number 8 and 10.

Example. Page 138 number 22.

Note. You are familiar with the concept of dimension: A line is 1-dimensional, a plane is 2-dimensional, the physical space we live in is 3-dimensional, and the spacetime of relativity is 4-dimensional. We associate a dimension with a vector space, but we need some technical results before we can give a definition.

Theorem 2.2. Relative Sizes of Spanning and Independent Sets.

Let W be a subspace of $\mathbb{R}^n$. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be vectors in W that span W and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in W that are independent. Then $k \geq m$.

Corollary 2.1.A. Invariance of Dimension.

Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

Note. Now that we know that all bases of a given subspace of $\mathbb{R}^n$ are of the same size, then we can give this common parameter a name.

Definition 2.2. Let W be a subspace of $\mathbb{R}^n$. The number of elements in a basis for W is the *dimension* of W, denoted $\dim(W)$.

Note. The standard basis $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ of $\mathbb{R}^n$ has n vectors, so $\dim(\mathbb{R}^n) = n$.

Note. We now give some properties of bases of a subspace of $\mathbb{R}^n$. In particular, notice in the following that every subspace of $\mathbb{R}^n$ has a basis.

Theorem 2.3. Existence and Determination of Bases.

(1) Every subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

(2) Every independent set of vectors in $\mathbb{R}^n$ can be enlarged to become a basis of $\mathbb{R}^n$.

(3) If W is a subspace of $\mathbb{R}^n$ and $\dim(W) = k$ then

- **(a)** every independent set of k vectors in W is a basis for W, and
- **(b)** every set of k vectors in W that spans W is a basis of W.

Examples. Page 136 numbers 34 and 38.

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