Chapter 2. Dimension, Rank, and Linear Transformations

2.2. The Rank of a Matrix

Note. In this section, we consider the relationship between the dimensions of the column space, row space and nullspace of a matrix A. We discuss their dimensions and bases.

Theorem 2.4. Row Rank Equals Column Rank.
Let A be an $m \times n$ matrix. The dimension of the row space of A equals the dimension of the column space of A. The common dimension is the *rank* of A.

Note. Theorem 2.4 is a fundamental result concerning matrices. Its proof is rather involved. Fraleigh and Beauregard give an example illustrating the theorem and claim that it can be generalized. For a detailed argument, see my online notes for Theory of Matrices (MATH 5090): `http://faculty.etsu.edu/gardnerr/5090/notes/Chapter-3-3.pdf` (notice Theorem 3.3.2).

Note. We see from Theorem 2.1.A, "Finding a Basis for $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$," that the dimension of the column space of A is the number of pivots of a row-echelon form of A. So the rank of A is the number of pivot containing columns in H where

$A \sim H$ and H is in row-echelon form. In addition, Theorem 2.1.A shows that a basis for the column space of A is given by the columns of A corresponding to the columns of H which contain pivots. Since the row operations performed on A in reducing it to H do not change the row space of A (row operations correspond to various linear combinations and rearrangement of the rows of A), a basis for the row space of A is given by the nonzero rows of H. We summarize these observations now.

Note 2.2.A. Finding Bases for Spaces Associated with a Matrix.
Let A be an $m \times n$ matrix with row-echelon form H.
(1) for a basis of the row space of A, use the nonzero rows of H,
(2) for a basis of the column space of A, use the columns of A corresponding to the columns of H which contain pivots, and
(3) for a basis of the nullspace of A use H to solve $H\vec{x} = \vec{0}$ as before.

Example. Page 140 number 6.

Note. For $A \sim H$ where H is in row-echelon form, the number of pivot containing columns of H is the rank of A. When we perform the same row reduction for the augmented matrix for $A\vec{x} = \vec{0}$ we get $[A \mid \vec{0}] \sim [H \mid \vec{0}]$ and each pivot-free column of H corresponds to a free variable in the system of equations (see Theorem 1.7(3), "Solutions of $A\vec{x} = \vec{b}$") and each free variable corresponds to a basis vector of the nullspace of A. Hence the dimension of the nullspace of A is the number of pivot-free columns of H. We therefore have the following.

Theorem 2.5. Rank Equation.

Let A be $m \times n$ with row-echelon form H.

(1) The dimension of the nullspace of A is

$$\begin{aligned} \text{nullity}(A) &= (\#\text{ free variables in solution of } A\vec{x} = \vec{0}) \\ &= (\#\text{ pivot-free columns of } H). \end{aligned}$$

(2) $\text{rank}(A) = (\#\text{ of pivots in } H)$.

(3) Rank Equation: $\text{rank}(A) + \text{nullity}(A) = \#$ of columns of A.

Note. If A is square, say A is $n \times n$, and the rank of A is n then for $A \sim H$ where H is in reduced row echelon form implies that $H = \mathcal{I}$. That is, A is row equivalent to $\mathcal{I}$. By Theorem 1.12, "Conditions for A^{-1} to Exist," this implies that A is invertible. We can therefore add another condition to our collection which is equivalent to A being invertible (see also Theorem 1.12 of Section 1.5 and Theorem 1.16 of Section 1.6).

Theorem 2.6. An Invertibility Criterion.

An $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$.

Example. Page 141 number 12. If A is square, then $\text{nullity}(A) = \text{nullity}(A^T)$.

Proof. The column space of A is the same as the row space of A^T, so $\text{rank}(A) = \text{rank}(A^T)$ and since the number of columns of A equals the number of columns of A^T, then by the Rank Equation:

$$\text{rank}(A) + \text{nullity}(A) = \text{rank}(A^T) + \text{nullity}(A^T)$$

and the result follows. ■

Examples. Page 141 numbers 14 and 18.

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