Chapter 2. Dimension, Rank, and Linear Transformations

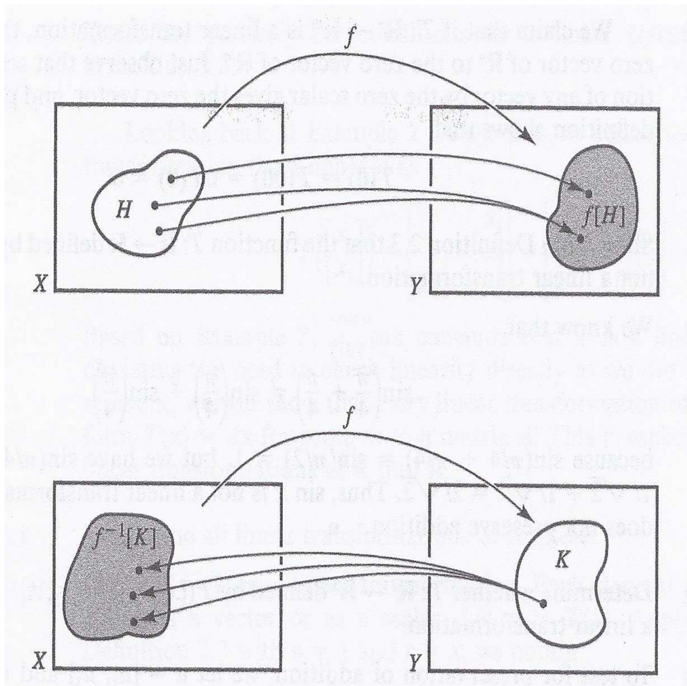
2.3 Linear Transformations of Euclidean Spaces

Note. We start with another quote from Fraleigh and Beauregard (see page 142):

> "Functions are used throughout mathematics to study the structures of sets and relationships between sets. You are familiar with the notation $y = f(x)$, where f is a function that acts on numbers signified by the input variable x, and produces numbers signified by the output variable y. In linear algebra, we are interested in functions $\vec{y} = f(\vec{x})$, where f acts on vectors, signified by the input variable $\vec{x}$, and produces vectors signified by the output variable $\vec{y}$."

Definition. Let X and Y be sets. A *function* $f : X \to Y$ is a rule that associates with each $x \in X$ an element $y \in Y$, denoted $y = f(x)$. (We read "$f : X \to Y$" as "f maps X into Y.") Set X is the *domain* of f and set Y is the *codomain* of f. For $H \subset X$ ("H a subset of X"), define $f[H] = \{f(h) \mid h \in H\}$; $f[H]$ is the *image* of H under f. The image of domain X under f, $f[X]$, is the *range* of f. For $K \subset Y$, the set $f^{-1}[K] = \{x \in K \mid f(x) \in K\}$ is the *inverse image* of K under f.

Note. See Figure 2.2 for an illustration of an image and inverse image. Notice that an inverse image is defined even if f is not invertible (that is, even if f is not one-to-one).



Note. We are interested in "linear transformations" mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ (after all, this is *linear* algebra!). We'll see that such transformations are associated with matrices (and conversely), justifying our earlier exploration of matrices.

Definition 2.3. A *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function whose *domain* is $\mathbb{R}^n$ and whose *codomain* is $\mathbb{R}^m$, where

(1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$, and

(2) $T(r\vec{u}) = rT(\vec{u})$ for all $\vec{u} \in \mathbb{R}^n$ and for all $r \in \mathbb{R}$.

Example. Page 153 Number 32. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that

$$T(r\vec{u} + s\vec{v}) = rT(\vec{u}) + sT(\vec{v})$$

for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and r and s. (As the text says, "linear transformations preserve linear combinations.")

Note. An immediate property of a linear transformation is that it maps zero vectors to zero vectors:

$$\begin{aligned} T(\vec{0}) &= T(0\vec{0}) \text{ since } 0\vec{v} = \vec{0} \text{ for any vector } \vec{v} \\ &= 0T(\vec{0}) \text{ by Definition 2.3(2), ``Linear Transformation''} \\ &= \vec{0}. \end{aligned}$$

Example. Page 144 Example 3. Let A be an $m \times n$ matrix and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T_A(\vec{x}) = A\vec{x}$ for each column vector $\vec{x} \in \mathbb{R}^n$. Prove that T_A is a linear transformation.

Note. The previous example shows that matrix multiplication is an example of linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$. We'll see later (in Corollary 2.3.A) that the converse holds and that every linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is represented by a matrix.

Example. Page 152 number 4.

Example. Page 145 Example 4. Notice that every linear transformation of $\mathbb{R} \to \mathbb{R}$ is of the form

$$T([x]) = [mx].$$

The graphs of such functions (with vectors in standard position) are lines through the origin.

Note. The following result shows that a linear transformation mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ is determined by its values on the elements of a basis of $\mathbb{R}^n$. This shouldn't be surprising since each element of $\mathbb{R}^n$ is a linear combination of basis vectors.

Theorem 2.7. Bases and Linear Transformations.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $B = \{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}\}$ be a basis for $\mathbb{R}^n$. For any vector $\vec{v} \in \mathbb{R}^n$, the vector $T(\vec{v})$ is uniquely determined by $T(\vec{b_1}), T(\vec{b_2}), \ldots, T(\vec{b_n})$.

Corollary 2.3.A. Standard Matrix Representation of Linear Transformations.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, and let A be the $m \times n$ matrix whose jth column is $T(\hat{e_j})$. Then $T(\vec{x}) = A\vec{x}$ for each $\vec{x} \in \mathbb{R}^n$. A is the *standard matrix representation* of T.

Note. Combining Example 3 and Corollary 2.3.A, we see that every linear combination T mapping $\mathbb{R}^n \to \mathbb{R}^m$ corresponds to some $m \times n$ matrix A (namely, the standard matrix representation of T), and conversely every matrix A corresponds to a linear transformation (namely T_A defined as $T_A(\vec{x}) = A\vec{x}$).

Example. Page 152 number 10.

Note. Since we have a one-to-one correspondence between linear transformations mapping $\mathbb{R}^n \to \mathbb{R}^m$ and $m \times n$ matrices, we can extend many of the ideas about matrices to linear transformations.

Definition. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix representation A. The *kernel* of T is the nullspace of A, denoted $\ker(T)$.

Theorem 2.3.A. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix representation A.

(1) The *range* $T[\mathbb{R}^n]$ of T is the column space of A.

(2) If W is a subspace of $\mathbb{R}^n$, then $T[W]$ is a subspace of $\mathbb{R}^m$ (i.e. T preserves subspaces).

Note 2.3.A. If A is the standard matrix representation for T, then from Theorem 2.5, "Rank Equation," we have:

$$\dim(\text{range } T) + \dim(\ker T) = \dim(\text{domain } T).$$

Definition. For a linear transformation T, we define *rank* and *nullity* as follows:

$$\text{rank}(T) = \dim(\text{range } T), \qquad \text{nullity}(T) = \dim(\ker T).$$

Definition. If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $T' : \mathbb{R}^m \to \mathbb{R}^k$, then the *composition* of T and T' is $(T' \circ T) : \mathbb{R}^n \to \mathbb{R}^k$ where $(T' \circ T)\vec{x} = T'(T(\vec{x}))$.

Note. In Exercise 31, it is shown that a composition of linear transformations is linear. Therefore a composition of linear transformations has a standard matrix representation by Corollary 2.3.A. The following result relates the standard matrix representation of a composition of linear transformations to the standard matrices of the constituent linear transformations.

Theorem 2.3.B. Matrix Multiplication and Composite Transformations. A composition of two linear transformations T and T' with standard matrix representation A and A' yields a linear transformation $T' \circ T$ with standard matrix representation $A'A$.

Example. Page 153 number 20.

Note. Since we have defined a composition of linear transformations (provided the codomain of the first transformation equals the domain of the second linear transformation) then we can define the inverse of a linear transformation. However, as with matrices, inverses may not always exist.

Definition. If $T : \mathbb{R}^n \to \mathbb{R}^n$ and there exists $T' : \mathbb{R}^n \to \mathbb{R}^n$ such that $T \circ T'(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$, then T' is the *inverse* of T denoted $T' = T^{-1}$. (Notice that if $T : \mathbb{R}^m \to \mathbb{R}^n$ where $m \neq n$, then T^{-1} is not defined; there are domain/codomain problems.)

Note. The following result follows easily from Theorem 2.3.B.

Theorem 2.3.C. Invertible Matrices and Inverse Transformations.
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ have standard matrix representation A: $T(\vec{x}) = A\vec{x}$. Then T is invertible if and only if A is invertible and $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

Example. Page 153 Number 23.

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