Chapter 2. Dimension, Rank, and Linear Transformations

2.4 Linear Transformations of the Plane

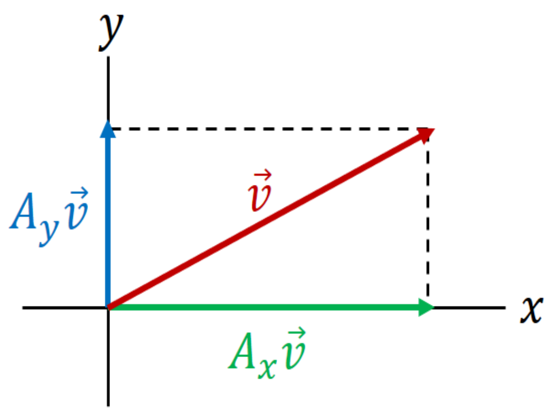
Note. By Corollary 2.3.A, "Standard Matrix Representation of Linear Transformations," we know that every linear transformation form $\mathbb{R}^2$ to $\mathbb{R}^2$ is represented by a 2×2 matrix. In this section we describe various types of such transformations and classify invertible transformations of $\mathbb{R}^2$ in terms of certain elementary types of transformations.

Note. If A is a 2×2 matrix with rank 0 then it is the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The linear transformation which is represented by A "collapses" all of $\mathbb{R}^2$ down to the zero vector.

Note. If a 2×2 matrix A is of rank 1 then either A has exactly one nonzero column or A has two nonzero columns and each is a multiple of the other. Two special cases are $A_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that $A_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $A_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$. Geometrically, A_x projects vectors onto the x-axis and A_y projects vectors onto the y-axis:



Note. If A is rank 1 and each column is a multiple of the other, then $A = \begin{bmatrix} a & ka \\ b & kb \end{bmatrix}$ for some constant k. Then

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & ka \\ b & kb \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + kay \\ ax + kby \end{bmatrix} = \begin{bmatrix} ax + kay \\ (b/a)(ax + kay) \end{bmatrix}.$$

Here, the second entry is b/a times the first entry (if $a \neq 0$). So the vector $A\begin{bmatrix} x \\ y \end{bmatrix}$ has *its* second entry as b/a times its first entry and so the vector (when in standard position) lies along the line $y = (b/a)x$. So $A\begin{bmatrix} x \\ y \end{bmatrix}$ geometrically collapses $\mathbb{R}^2$onto the line $y = (b/a)x$ (notice that this is independent of the value of k and only depends on the ratio of b to a). We will explore projections in more detail in Chapter 6; matrix A here does not represent a projection.

Note. If $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear transformation given by $T_A(\vec{x}) = A\vec{x}$ then A is an invertible 2×2 matrix by Theorem 2.3.A. We now consider some special invertible transformations.

Note. We now consider a *rotation* about the origin of a vector through an angle θ. To find the standard matrix representing such a rotation, we only need to consider rotations of $\hat{e}_1$ and $\hat{e}_2$:

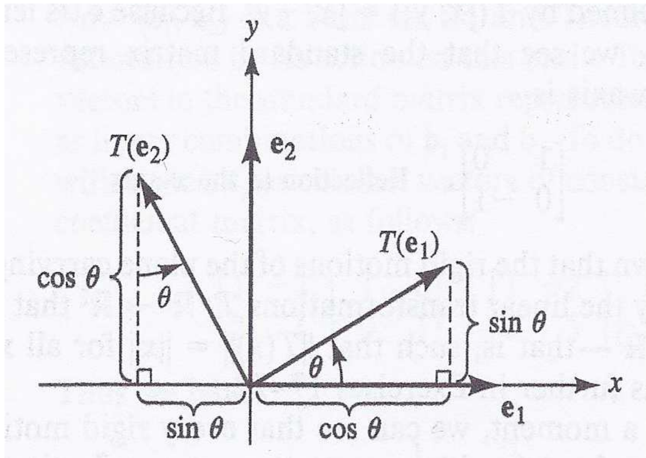
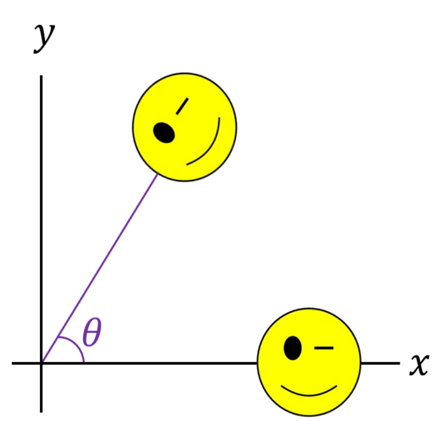


FIGURE 2.9
Counterclockwise rotation of e_1 and e_2 through the angle θ.

We see that $T(\hat{e}_1) = [\cos\theta, \sin\theta]$ and $T(\hat{e}_2) = [-\sin\theta, \cos\theta]$. So we have the standard matrix representation of such a rotation as

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This is an example of a *rigid* transformation of the plane since lengths are not changed under this transformation. We can illustrate it as follows:



Example. Page 165 Number 4.

Note. We can *reflect* a vector in $\mathbb{R}^2$ about the x-axis by applying T_X where

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

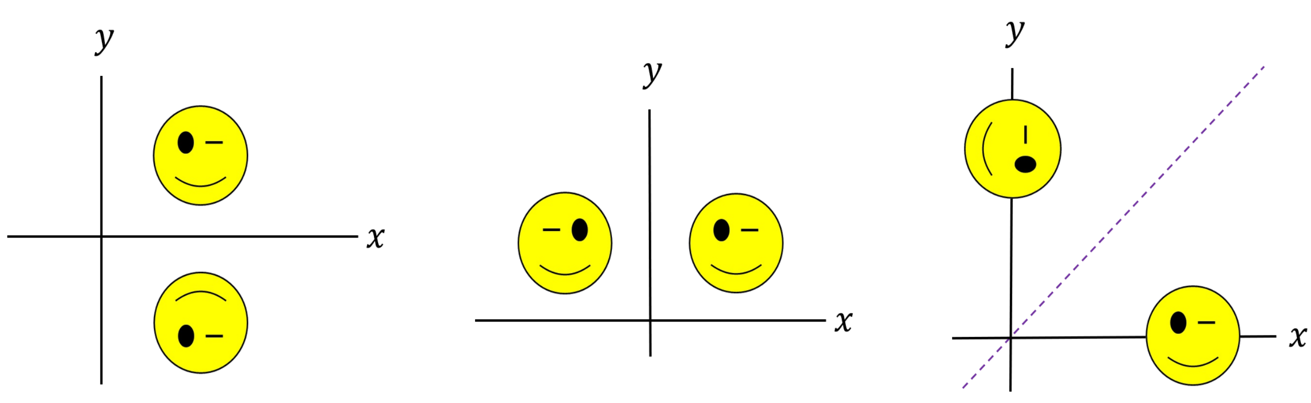
We can *reflect* a vector in $\mathbb{R}^2$ about the y-axis by applying T_Y where

$$Y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can *reflect* a vector in $\mathbb{R}^2$ about the line $y = x$ by applying T_Z where

$$Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that X, Y, and Z are elementary matrices since they differ from $\mathcal{I}$ by an operation of row scaling (for X and Y), or by an operation of row interchange (for Z). Geometrically, we have:



X Y Z

Example. Page 165 Number 6.

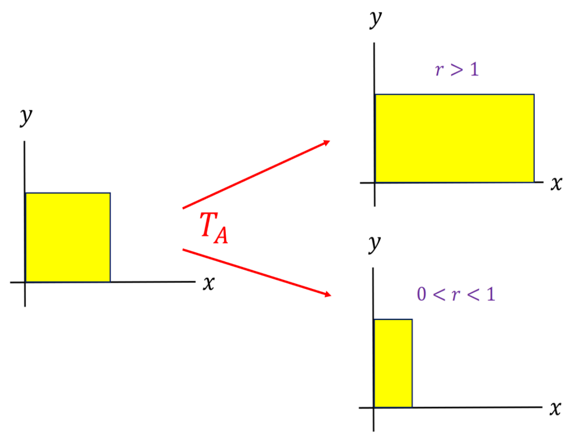
Note. Transformation T_A where

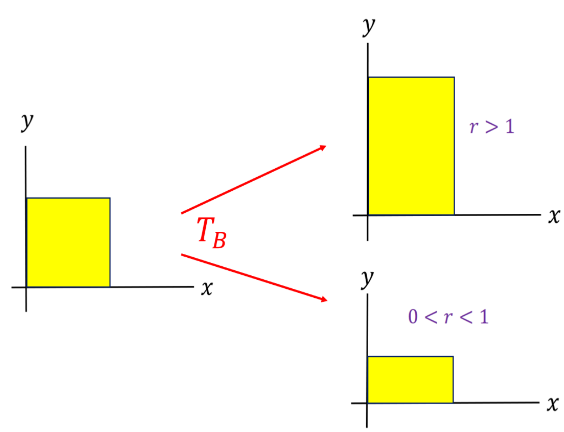
$$A = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$$

is a *horizontal expansion* if $r > 1$, and is a *horizontal contraction* if $0 < r < 1$. Transformation T_B where

$$B = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$$

is a *vertical expansion* if $r > 1$, and is a *vertical contraction* if $0 < r < 1$. Notice that A and B are elementary matrices since they differ from $\mathcal{I}$ by an operation of row scaling.





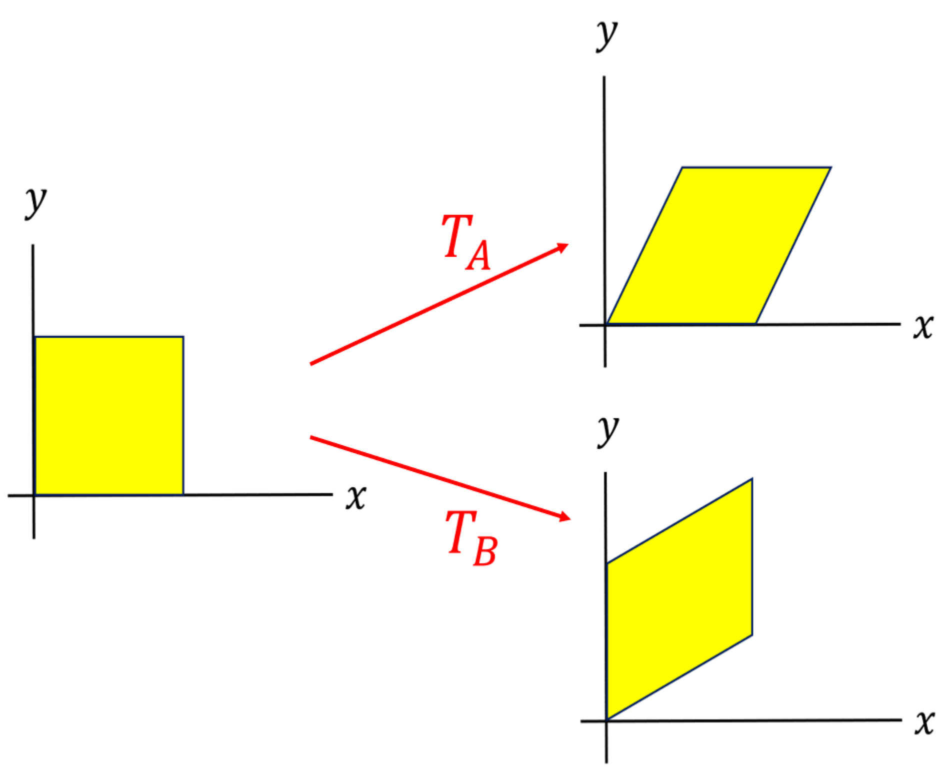
Note. Transformation T_A where

$$A = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$$

is a *horizontal shear*. Transformation T_B where

$$B = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

is a *vertical shear* (see Figure 2.2.16 on page 163). Notice that A and B are elementary matrices since they differ from $\mathcal{I}$ by an operation of row addition.



Example. Page 165 Number 8(iii, iv).

Theorem 2.4.A. Geometric Description of Invertible Transformations of $\mathbb{R}^2$.

A linear transformation T of the plane $\mathbb{R}^2$ into itself is invertible if and only if T consists of a finite sequence of:

- Reflections in the x-axis, the y-axis, or the line $y = x$;
- Vertical or horizontal expansions or contractions; and
- Vertical or horizontal shears.

Examples. Page 165 Number 14, Page 166 Number 18, Page 166 Number 20.

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