# Chapter 2. Dimension, Rank, and Linear Transformations

## 2.5 Lines, Planes, and Other Flats

**Note.** In this section, we continue with the geometric interpretation of vectors introduced in Section 1.1. We now consider a collection of vectors all placed in standard position. So the tails of the vectors are all placed at the origin of a Cartesian coordinate system and the heads of the vectors determine collections of points that form various geometric objects. First, we consider the situation where the points associated with the heads of the vectors determine a line.

**Definitions 2.4, 2.5.** Let $S$ be a subset of $\mathbb{R}^n$ and let $\vec{a} \in \mathbb{R}^n$. The set $\{\vec{x} + \vec{a} \mid \vec{x} \in S\}$ is the *translate* of $S$ by $\vec{a}$, and is denoted by $S + \vec{a}$. The vector $\vec{a}$ is the *translation vector*. A *line* in $\mathbb{R}^n$ is a translate of a one-dimensional subspace of $\mathbb{R}^n$.

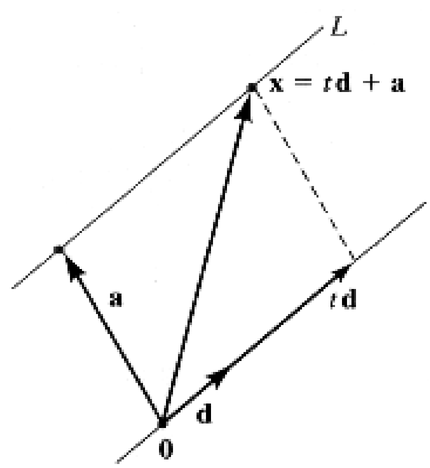


Figure 2.19, page 168.

**Definition.** If a line $L$ in $\mathbb{R}^n$ contains point $(a_1, a_2, \ldots, a_n)$ and if vector $\vec{d}$ is parallel to $L$, then $\vec{d}$ is a *direction vector* for $L$ and $\vec{a} = [a_1, a_2, \ldots, a_n]$ is a *translation vector* of $L$.

**Note.** With $\vec{d}$ as a direction vector and $\vec{a}$ as a translation vector of a line, we have $L = \{t\vec{d} + \vec{a} \mid t \in \mathbb{R}\}$. In this case, $t$ is called a *parameter* and we can express the line *parametrically* as a vector equation:

$$\vec{x} = t\vec{d} + \vec{a}$$

or as a collection of component equations:

$$\begin{aligned} x_1 &= td_1 + a_1 \\ x_2 &= td_2 + a_2 \\ &\vdots \\ x_n &= td_n + a_n. \end{aligned}$$

**Examples.** Page 176 Number 8, Page 176 Number 12(a).

**Note.** Now generalizing the concept of a line associated with a collection of vectors in standard position, we consider a plane associated with a collection of vectors in standard position. We can also consider a higher dimensional object associated with a collection of vectors in standard position.

**Definition 2.6.** A *k-flat* in $\mathbb{R}^n$ is a translate of a $k$-dimensional subspace of $\mathbb{R}^n$. In particular, a 1-flat is a *line*, a 2-flat is a *plane*, and an $(n-1)$-flat is a *hyperplane*. We consider each point of $\mathbb{R}^n$ to be a *zero-flat*.

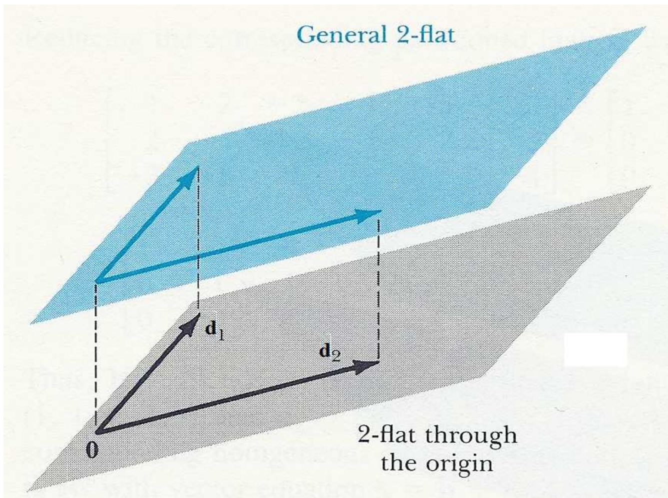


Figure 2.24, Page 171

**Note.** We can also talk about a translate of a $k$-dimensional subspace $W$ of $\mathbb{R}^n$. If a basis for $W$ is $\{\vec{d_1}, \vec{d_2}, \ldots, \vec{d_k}\}$, then the $k$-flat through the point $(a_1, a_2, \ldots, a_n)$ and parallel to $W$ is

$$\vec{x} = t_1\vec{d_1} + t_2\vec{d_2} + \cdots + t_k\vec{d_k} + \vec{a}$$

where $\vec{a} = [a_1, a_2, \ldots, a_n]$ and $t_1, t_2, \ldots, t_k \in \mathbb{R}$ are *parameters*. We can also express this $k$-flat parametrically in terms of components.

**Examples.** Page 177 Number 22, Page 177 Number 30.

**Note.** We can now clearly explain the geometric interpretation of solutions of linear systems in terms of $k$-flats. Consider $A\vec{x} = \vec{b}$, a system of $m$ equations in $n$ unknowns that has at least one solution $\vec{x} = \vec{p}$. By Theorem 1.18 on page 97, the solution set of the system consists of all vectors of the form $\vec{x} = \vec{p} + \vec{h}$ where $\vec{h}$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$. Now the solution set of $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$, and so the solution of $A\vec{x} = \vec{b}$ is a $k$-flat (where $k$ is the nullity of $A$) passing through point $(p_1, p_2, \ldots, p_n)$ where $\vec{p} = [p_1, p_2, \ldots, p_n]$.

**Example.** Page 177 Number 36.

*Revised: 2/22/2018*