Chapter 3. Vector Spaces

Note. We quote Fraleigh and Beauregard's insightful first paragraph in Chapter 3 from page 179:

> "For the sake of efficiency, mathematicians often study objects just in terms of their mathematical structure, deemphasizing such things as particular symbols used, names of things, and applications. Any properties derived exclusively from mathematical structure will hold for all objects having that structure. Organizing mathematics in this way avoids repeating the same arguments in different contexts. Viewed from this perspective, linear algebra is the study of all objects that have a *vector-space* structure. The Euclidean spaces $\mathbb{R}^n$ that we treated in Chapters 1 and 2 serve as our guide."

So vector spaces are the overarching concept for this class!

3.1 Vector Spaces

Note. In the first part of Chapter 1 we introduced vectors in $\mathbb{R}^n$ and studied some of their properties (norms, dot products, bases, etc.). We now define a general abstract mathematical structure called a "vector space." We are motivated by properties of $\mathbb{R}^n$ in this definition and $\mathbb{R}^n$ is an *example* of a vector space, though there are many other examples of vector spaces. In fact, some of the most useful applications of vector spaces involve spaces where the vectors are functions.

Definition 3.1. A *vector space* is a set V of *vectors* along with an operation of addition $+$ of vectors and multiplication of a vector by a scalar (real number), which satisfies the following. For all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all $r, s \in \mathbb{R}$:

A1. Associativity of vector addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

A2. Commutivity of vector addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

A3. Vector additive identity: There exists $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v}$

A4. Vector additive inverse: $\vec{v} + (-\vec{v}) = \vec{0}$

S1. Distribution of scalar multiplication over vector addition: $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$

S2. Distribution of scalar multiplication over scalar addition: $(r + s)\vec{v} = r\vec{v} + s\vec{v}$

S3. Associativity of scalars: $r(s\vec{v}) = (rs)\vec{v}$

S4. Preservation of scale (or scalar multiplicative identity): $1\vec{v} = \vec{v}$

Definition. $\vec{0}$ is the *additive identity.* $-\vec{v}$ is the *additive inverse* of $\vec{v}$.

Note. Of course $\mathbb{R}^n$ is an example of a vector space for each natural number n. In Example 3.1.1 it is argued that the set $M_{m,n}$ of all $m \times n$ matrices is a vector space where the matrices are vectors and addition and scalar multiplication are as given in Section 1.3.

Example 3.1.2. The set $\mathcal{P}$ of all polynomials in variable x with real coefficients is a vector space. Vector addition and scalar multiplication are the usual addition of polynomials and multiplication of a polynomial by a scalar.

Note. In Exercise 3.1.16, it is to be shown that the set $\mathcal{P}_n$ of all polynomials in variable x with real coefficients and of degree less than or equal to n (together with the zero polynomial) is a vector space.

Example 3.1.3. Let $\mathcal{F}$ be the set of all real-valued functions with domain $\mathbb{R}$. The vector sum of f and g in $\mathcal{F}$ is defined as $(f+g)(x) = f(x)+g(x)$. For scalar $r \in \mathbb{R}$, scalar multiplication is defined as $(rf)(x) = rf(x)$.

Partial Solution. The zero vector is just the constant function $0(x) = 0$. For $f \in \mathcal{F}$ we have $-f = (-1)f$. We can then verify A1–A4 and S1–S4 similar to above (it is actually less tedious in this example). $\square$

Page 189 Number 6. Consider the set $\mathcal{F}$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, with scalar multiplication defined for scalar $r \in \mathbb{R}$ and $f \in \mathcal{F}$ as $(rf)(x) = rf(x)$, and vector addition $\tilde{+}$ defined as $(f \,\tilde{+}\, g)(x) = \max\{f(x), g(x)\}$. Is $\mathcal{F}$ a vector space?

Example 3.1.A. Another, much more exotic, example of a vector space is the set of all functions f with domain $[0,1]$ such that $\displaystyle\int_0^1 f(x)^2\,dx < \infty$. This vector space is denoted $L^2[0,1]$:

$$L^2[0,1] = \left\{ f \,\middle|\, \int_0^1 f(x)^2\,dx < \infty \right\}.$$

Many of the questions in applied math (and quantum theory) can be translated into questions in this space. This vector space is a "Hilbert space" and is studied in upper level analysis classes.

Theorem 3.1. Elementary Properties of Vector Spaces.

Every vector space V satisfies:

1. the vector $\vec{0}$ is the unique additive identity in a vector space,

2. for each $\vec{v} \in V$, $-\vec{v}$ is the unique additive inverse of $\vec{v}$,

3. if $\vec{u} + \vec{v} = \vec{u} + \vec{w}$ then $\vec{v} = \vec{w}$,

4. $0\vec{v} = \vec{0}$ for all $\vec{v} \in V$,

5. $r\vec{0} = \vec{0}$ for all scalars $r \in \mathbb{R}$,

6. $(-r)\vec{v} = r(-\vec{v}) = -(r\vec{v})$ for all $r \in \mathbb{R}$ and for all $\vec{v} \in V$.

Note. Proofs of parts 1 and 4 are given in the text on pages 185 and 186. Proofs of parts 2 (page 190 Number 19), 5 (Page 190 Number 21), and 6 (Page 190 Number 22) are left as exercises. We now give proofs of parts 1 and 3.

Page 190 Number 24. Let V be a vector space and let $\vec{v}$ and $\vec{w}$ be nonzero vectors in V. Prove that if $\vec{v}$ is not a scalar multiple of $\vec{w}$, then $\vec{v}$ is not a scalar multiple of $\vec{v} + \vec{w}$.

Note. The text introduces the "Universality of Function Spaces" in which the function space of Example 3.1.3 is modified from the set of all real-valued functions with domain $\mathbb{R}$ to the set of all real-valued functions with domain a given *set*. For example, if set $S = \{1, 2, \ldots, n\}$, then a given function $f : S \to \mathbb{R}$ yields n real values, $f(1)$, $f(2)$, $\ldots$, $f(n)$. We then have that the set $\mathcal{F}$ of all such functions form a vector space (with all the same basic argument as used in Example 3.1.3).

In fact, if we associate the vector $[f(1), f(2), \ldots, f(n)]$ in $\mathbb{R}^n$ with vector $f \in \mathcal{F}$, then we see that the vector space "structure" of $\mathcal{F}$ and $\mathbb{R}^n$ are identical. We'll elaborate on this idea in Section 3.4 when we introduce vector space isomorphisms.

Note. We can also use the Universality of Function Spaces in connection with the vector space of all $m \times n$ matrices, $M_{m,n}$. Let set $S = \{(1,1), (1,2), \ldots, (1,n), (2,1), (2,2), \ldots, (2,n), (3,1), (3,2), \ldots, (m-1,n), (m,1), (m,2), \ldots, (m,n)\}$. Then for $f : S \to \mathbb{R}$ we have a "natural relationship" between f and an $m \times n$ matrix

$$M_f = \begin{bmatrix} f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\ f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\ \vdots & \vdots & \ddots & \vdots \\ f((m,1)) & f((m,2)) & \cdots & f((m,n)) \end{bmatrix}.$$

Page 190 Number 26. Use the universality of function spaces to explain how we can view the Euclidean vector space $\mathbb{R}^{mn}$ and the vector space $M_{m,n}$ of all $m \times n$ matrices as essentially the same vector space with just a different notation for the vectors.

Revised: 6/30/2019