Chapter 3. Vector Spaces

3.2 Basic Concepts of Vector Spaces

Note. We now extend the ideas developed for $\mathbb{R}^n$ in Chapters 1 and 2, such as linear combination, span, subspace, basis, and dimension, to the setting of general vector spaces. In the process, we introduce some new examples of vector spaces.

Definition 3.2. Given vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in V$ and scalars $r_1, r_2, \ldots, r_k \in \mathbb{R}$,

$$\sum_{\ell=1}^{k} r_\ell \vec{v_\ell} = r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_k\vec{v_k}$$

is a *linear combination* of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ with *scalar coefficients* $r_1, r_2, \ldots, r_k$.

Definition 3.3. Let X be a subset of vector space V. The *span* of X is the set of all linear combinations of elements in X and is denoted $\text{sp}(X)$. If $V = \text{sp}(X)$ for some finite set X, then V is *finitely generated.*

Note. Let $\mathcal{P}$ be the vector space of all polynomials with real coefficients (see Example 3.1.2). Let $M = \{1, x, x^2, x^3, \ldots\} \subset \mathcal{P}$. Then $\mathcal{P} = \text{sp}(M)$ since any $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is clearly in $\text{sp}(M)$. Notice also that if we exclude any element of M, say x^m, then we no longer have a spanning set of $\mathcal{P}$ since the polynomial $q(x) = x^m$ is not in the span of the modified set. So $\mathcal{P}$ is not finitely generated.

Definition 3.4. A subset W of a vector space V is a *subspace* of V if W is itself a vector space.

Theorem 3.2. Test for Subspace.

A subset W of vector space V is a subspace if and only if

(1) $\vec{v}, \vec{w} \in W \Rightarrow \vec{v} + \vec{w} \in W$,

(2) for all $r \in \mathbb{R}$ and for all $\vec{v} \in W$, we have $r\vec{v} \in W$.

Note. Two obvious subspaces of any vector space V are $\{\vec{0}\}$ and V itself. Any subspace of V which is not V itself is a *proper subspace* of V. Subspace $\{\vec{0}\}$ is the *zero subspace* (or *trivial subspace*).

Note. In the vector space $M_n = M_{n,n}$ of all $n \times n$ matrices, the set U of all upper-triangular matrices forms a subspace since the sum of any two upper-triangular matrices is upper triangular and a scalar multiple of an upper-triangular matrix is upper triangular.

Note 3.2.A. In the vector space $\mathcal{F}$ of all real-valued functions with domain $\mathbb{R}$ (see Example 3.1.3), some subspaces are:

$$\begin{aligned} C &= \{f \in \mathcal{F} \mid f \text{ is continuous}\} \\ D &= \{f \in \mathcal{F} \mid f \text{ is differentiable}\} \\ D_n &= \{f \in \mathcal{F} \mid f \text{ is } n\text{-times differentiable}\} \\ D_\infty &= \{f \in \mathcal{F} \mid f \text{ is differentiable of all orders}\} \end{aligned}$$

In fact, D is a subspace of C (since all differentiable functions are continuous), D_n is a subspace of D for all n, and D_∞ is a subspace of D_n for all n.

Examples. Page 202 number 4, Page 202 Number 8.

Definition 3.5. Let X be a set of vectors from a vector space V. A *dependence relation* in X is an equation of the form

$$\sum_{\ell=1}^{k} r_\ell \vec{v_\ell} = r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_k\vec{v_k} = \vec{0}$$

with some $r_j \neq 0$ and $\vec{v_i} \in X$. If such a relation exists, then X is a *linearly dependent* set. Otherwise X is a *linearly independent* set.

Example. Page 202 number 16.

Definition 3.6. Let V be a vector space. A set of vectors in V is a *basis* for V if

(1) the set of vectors span V, and

(2) the set of vectors is linearly independent.

Note 3.2.B. Consider the vector space $\mathcal{P}_n$ of all polynomials with real coefficients of degree less than or equal to n (see Exercise 3.1.16). A basis is given by $B = \{1, x, x^2, \ldots, x^n\}$. This follows because any $p(x) \in \mathcal{P}_n$ is of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and so clearly $p(x) \in \mathrm{sp}(1, x, x^2, \ldots, x^n)$. The zero vector in $\mathcal{P}_n$ is $0 = 0x^n + 0x^{n-1} + \cdots + 0x + 0$, so $r_1 1 + r_2 x + r_3 x^2 + \cdots + r_{n+1}x^n = 0$ only for $r_1 = r_2 = \cdots = r_{n+1} = 0$, and B is a linearly independent set.

Examples. Page 202 number 20, Page 202 Number 22.

Theorem 3.3. Unique Combination Criterion for a Basis.
Let B be a set of nonzero vectors in vector space V. Then B is a basis for V if and only if each vector V can by uniquely expressed as a linear combination of the vectors in set B.

Note. The proof of Theorem 3.3 is complicated by the fact that a basis could be infinite. This was not an issue with the corresponding result for a subspace of $\mathbb{R}^n$ (Theorem 2.1, "Alternative Characterization of a Basis") since we could put our hands on the finite basis in that case and use all basis elements in any linear combination. The next two results (Theorem 3.4, "Relative Size of Spanning and Independent Sets" and Corollary 3.2.A) deal with finite dimensional vector spaces and so the proofs in these cases are identical to the proofs for $\mathbb{R}^n$ (Theorem 2.2 and Corollary 2.1.A).

Definition. A vector space is *finitely generated* if it is the span of some finite set.

Note. As long as we only consider finite dimensional vector spaces (which is the topic of this class), we need not concern ourselves with the subtleties of infinite dimensional vector spaces. For completeness, we comment that one can prove that every vector space actually has a basis (notice that the definition of "vector space" does not address the existence of a basis). Unfortunately, the existence proof requires the "Axiom of Choice" from set theory (see the footnote on page 198). An implication of this is that we know a basis exists but we have no idea which vectors make up the basis. This makes the result useless as far as applied math goes, but it is definitely of a theoretical interest.

Theorem 3.4. Relative Size of Spanning and Independent Sets.

Let V be a vector space. Let $\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k}$ be vectors in V that span V and let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ be vectors in V that are independent. Then $k \geq m$.

Corollary 3.2.A. Invariance of Dimension for Finitely Generated Spaces.

Let V be a finitely generated vector space. Then any two bases of V have the same number of elements.

Note. As in Section 2.1, now that we know that all bases of a finite dimensional vector space are of the same size, we can give this common parameter a name.

Definition 3.7. Let V be a finitely generated vector space. The number of elements in a basis for V is the *dimension* of V, denoted $\dim(V)$.

Note 3.2.C. Since $B = \{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{P}_n$ (see Note 3.2.B), the vector space of all polynomials of degree less than or equal to n, then $\dim(\mathcal{P}_n) = n + 1$.

Examples. Page 203 Number 32, Page 203 Number 36.

Note. In the proof given in Exercise 3.2.36, we never actually used the fact that $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ is linearly independent, but only used the fact that it is a spanning set for W. We can therefore conclude:

> **Corollary 3.2.B.** In n-dimensional vector space V, any linearly independent set of n vectors is a basis for V.

Example. Page 204 Number 40.

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