Chapter 3. Vector Spaces

3.3 Coordinatization of Vectors

Note. In Chapters 1 and 2 we studied the vector space $\mathbb{R}^n$. In Sections 3.1 and 3.2 we defined general vector spaces and studied some of their properties. In this section we "associate" a general dimension n vector space with the space $\mathbb{R}^n$. This allows us to study properties of general vector spaces using our knowledge of $\mathbb{R}^n$ and the use of matrix techniques in addressing properties of $\mathbb{R}^n$.

Note. We now define an ordered basis for a finite dimensional vector space and use the coefficients of these basis elements (in order) when expressing a given vector as a linear combination of the basis elements to associate a vector in $\mathbb{R}^n$ with the given vector.

Definition. An *ordered basis* $(\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n})$ is an "ordered set" of vectors which is a basis for some vector space.

Definition 3.8. If $B = (\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n})$ is an ordered basis for V and $\vec{v} = r_1\vec{b_1} + r_2\vec{b_2} + \cdots + r_n\vec{b_n}$, then the vector $[r_1, r_2, \ldots, r_n] \in \mathbb{R}^n$ is the *coordinate vector of* $\vec{v}$ *relative to* B, denoted $\vec{v_B}$.

Example 3.3.A. Let $\mathcal{P}_n$ be the vector space of all polynomials with real coefficients and of degree n or less (see Exercise 3.1.16). Then two ordered bases of $\mathcal{P}_n$ are $B = (1, x, x^2, \ldots, x^n)$ and $B' = (x^n, x^{n-1}, \ldots, x, 1)$. When $n = 4$ and $p(x) = -x + x^3 + 2x^4$ the coordinate vectors relative to the bases are $p(x)_B = [0, -1, 0, 1, 2]$ and $p(x)_{B'} = [2, 1, 0, -1, 0]$.

Example. Page 211 Number 6.

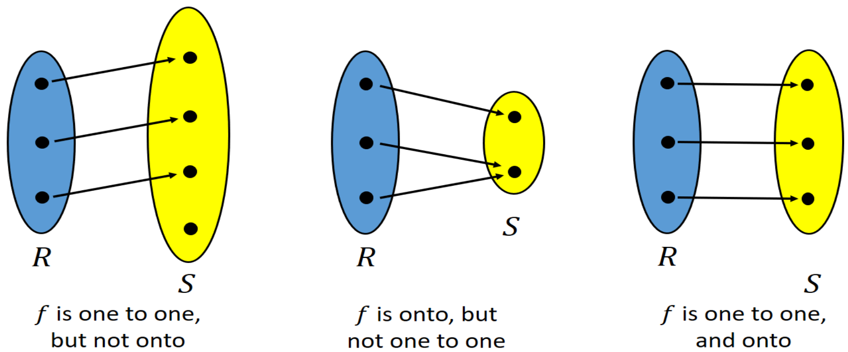
Note. We see from the solution of the previous example that the following algorithm applies in finding $\vec{v}_B$ when dealing with an ordered basis in $\mathbb{R}^n$.

Note 3.3.A. If the basis vectors in ordered basis B are in $\mathbb{R}^n$, then to find $\vec{v_B}$:
(1) write the basis vectors as column vectors to form $[\vec{b_1}\ \vec{b_2}\ \cdots\ \vec{b_n} \mid \vec{v}]$,
(2) use Gauss-Jordan elimination to get $[\mathcal{I} \mid \vec{v_B}]$.

Note. We are about to define the equipment for the most central result in this class! It will show that when dealing with an n-dimensional vector space, we may as well be dealing with $\mathbb{R}^n$. This is good news since we developed equipment in Chapters 1 and 2 to deal with questions involving $\mathbb{R}^n$ (namely, the matrix methods of these chapters).

Definition. Let f be a function mapping one set R to another set S, $f : R \to S$. Then f is *one to one* if $f(r_1) = f(r_2)$ implies that $r_1 = r_2$. Function f is *onto* if for each $s \in S$, there is some $r \in R$ such that $f(r) = s$.

Note. We can illustrate the previous definition as follows:



Definition. An *isomorphism* between two vector spaces V and W is a one-to-one and onto function α from V to W such that:

(1) if $\vec{v_1}, \vec{v_2} \in V$ then $\alpha(\vec{v_1} + \vec{v_2}) = \alpha(\vec{v_1}) + \alpha(\vec{v_2})$, and

(2) if $\vec{v} \in V$ and $r \in \mathbb{R}$ then $\alpha(r\vec{v}) = r\alpha(\vec{v})$.

If there is such an α, then V and W are *isomorphic*, denoted $V \cong W$.

Note. An isomorphism is a one-to-one and onto linear transformation, where by "linear" we mean a condition similar to that given in Definition 2.3, "Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$."

Note. The idea behind a vector space isomorphism is that two isomorphic vector spaces have the same structure. For example, $\mathbb{R}^3$ and $\mathcal{P}_2$ are both vector spaces of dimension 3 and they have the same structure; we just associate $\vec{v} = [a_2, a_1, a_0] \in \mathbb{R}^3$ with $p(x) = a_2x^2 + a_1x + a_0$ (and conversely). Since addition of $\vec{v}, \vec{w} \in \mathbb{R}^3$ where $\vec{v} = [a_2, a_1, a_0]$ and $\vec{w} = [b_2, b_1, b_0]$ corresponds to addition of $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0$, because $\vec{v} + \vec{w} = [a_2 + b_2, a_1 + b_1, a_0 + b_0]$ and $p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$, and scalar multiplication of $\vec{v} \in \mathbb{R}^3$ corresponds to scalar multiplication of $p(x)$, because $r\vec{v} = [ra_2, ra_1, ra_0]$ and $rp(x) = (ra_1)x^2 + (ra_1)x + (ra_0)$, then the association of $\vec{v} = [a_2, a_1, a_0]$ and $p(x) = a_2x^2 + a_1x + a_0$ is an isomorphism.

Note. We now state a result so important that your humble instructor refers to it as the "Fundamental Theorem of Finite Dimensional Vector Spaces." It justifies our time spent on vector space $\mathbb{R}^n$ in Chapters 1 and 2 as time well-spent on a central concept and not simply as time spent on motivation for our current study of general vector spaces. Beware that this terminology "Fundamental Theorem of Finite Dimensional Vector Spaces" is not widespread! It was invented by your instructor and appears in *Real Analysis with an Introduction to Wavelets*, D. Hong, J. Wang, and R. Gardner, Academic Press/Elsevier Press, (2005). This book also includes, by the way, the "Fundamental Theorem of Infinite Dimensional Vector Spaces." Fraleigh and Beauregard state this as "Coordinatization of Finite-Dimensional Spaces" (Theorem 3.9 in Section 3.4).

Theorem 3.3.A. The Fundamental Theorem of Finite Dimensional Vectors Spaces.
If V is a finite dimensional vector space (say $\dim(V) = n$) then V is isomorphic to $\mathbb{R}^n$.

Example 3.3.B. Consider $\mathcal{P}_n$, the vector space of all polynomials of degree n or less (see Exercise 3.1.16). Since $\dim(\mathcal{P}_n) = n + 1$ (see Section 3.2), so $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$. Find an isomorphism and prove that it is an isomorphism.

Examples. Page 212 Number 12, Page 212 Number 20.

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