Chapter 3. Vector Spaces

3.4 Linear Transformations

Note. We have already studied linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$. Now we look at linear transformations from one general vector space to another. We'll see parallel behavior between these linear transformations and the matrix transformations of Section 2.3. In fact, we use ordered bases to associate matrices with linear transformations between general finite dimensional vector spaces.

Definition 3.9. A function T that maps a vector space V into a vector space V' is a *linear transformation* if it satisfies:
(1) $(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, and **(2)** $T(r\vec{u}) = rT(\vec{u})$,
for all vectors $\vec{u}, \vec{v} \in V$ and for all scalars $r \in \mathbb{R}$.

Note 3.4.A. Exercise 3.4.35 claims that the condition of $T : V \to V'$ being linear is equivalent to the condition $T(r\vec{u} + s\vec{v}) = rT(\vec{u}) + sT(\vec{v})$ for all $\vec{u}, \vec{v} \in V$ and for all $r, s \in \mathbb{R}$. Notice that the same claim was established for V and V' Euclidean spaces in Exercises 2.3.32. We can conclude (from Mathematical Induction, see Appendix A) that for $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ and $r_1, r_2, \ldots, r_n \in \mathbb{R}$, we have

$$T(r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n) = r_1T(\vec{v}_1) + r_2T(\vec{v}_2) + \cdots + r_nT(\vec{v}_n).$$

Example 3.4.A. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let a be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$. Is T a linear transformation?

Definition. For a linear transformation $T : V \to V'$, the set V is the *domain* of T and the set V' is the *codomain* of T. If W is a subset of V, then $T[W] = \{T(\vec{w}) \mid \vec{w} \in W\}$ is the *image* of W under T. $T[V]$ is the *range* of T. For $W' \subset V'$, $T^{-1}[W'] = \{\vec{v} \in V \mid T(\vec{v}) \in W'\}$ is the *inverse image* of W' under T. $T^{-1}[\{\vec{0'}\}]$ if the *kernel* of T, denoted $\ker(T)$. Notice that $\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = 0'\}$.

Example 3.4.B. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let a be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Describe the kernel of T.

Definition. Let V, V' and V'' be vector spaces and let $T : V \to V'$ and $T' : V' \to V''$ be linear transformations. The *composition transformation* $T' \circ T : V \to V''$ is defined by $(T' \circ T)(\vec{v}) = T'(T(\vec{v}))$ for $\vec{v} \in V$.

Example. Page 214 Example 1. Let $\mathcal{F}$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ (see Example 3.1.3), and let D be its subspace of all differentiable functions. Show that differentiation is a linear transformation of D into $\mathcal{F}$.

Example. Page 215 Example 3. Let $C_{a,b}$ be the set of all continuous functions mapping $[a, b] \to \mathbb{R}$. Then $C_{a,b}$ is a vector space (based on an argument similar to that which justifies that $C = \{f \in \mathcal{F} \mid f \text{ is continuous}\}$ is a subspace of $\mathcal{F}$, as mentioned in Note 3.2.A). Prove that $T : C_{a,b} \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\, dx$ is a linear transformation. Such a transformation which maps functions to real numbers is called a *linear functional*.

Example. Page 215 Example 4. Let C be the vector space of all continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Note 3.2.A). Let $a \in \mathbb{R}$ and let $T_a : C \to C$ be defined by $T_a(f) = \int_a^x f(t)\,dt$. Prove that T is a linear transformation.

Note. One might think that the differentiation operator $D : D \to \mathcal{F}$ and the operator $T_a : C \to C$ in the previous example are "inverses" of each other (we have not yet defined the inverse of a linear transformation from one general vector space to another). This is not the case, though, since $T_a(f) = \int_a^x f(t)\,dt$ implies that

$$T_a(f)(a) = \left(\int_a^x f(t)\,dt\right)\Big|_{x=a} = \int_a^a f(t)\,dt = 0,$$

so T_a maps continuous functions to continuous functions which are 0 at $x = a$. Now each $T_a(f)$ is differentiable since $\frac{d}{dx}[T_a(f)] = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$ by the Fundamental Theorem of Calculus. If we define $D_a = \{f \in D \mid d(a) = 0\}$ (D_a is a subspace of D based on an argument similar to that given in Exercise 3.2.4) then we have $T_a : C \to D_a$, $D : D_a \to C$, and for $f \in C$,

$$(D \circ T_a)(f) = D(T_a(f)) = D\left(\int_a^x f(t)\,dt\right) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) = f.$$

If $f \in D_a$ (so $f(a) = 0$) AND f is *continuously* differentiable (that is, f' is continuous) then

$$(T_a \circ D)(f) = T_a(D(f)) = T_a(f') = \int_a^x f'(t)\,dt = f(x) - f(a) = f(x) = f.$$

So if we define $D_{1,a} = \{f \in D_a \mid f' \text{ is continuous}\}$ (a subspace of D_a), then we do have that the differentiation $D : D_{1,a} \to C$ and $T_a : C \to D_{1,a}$ *are* inverses of each other.

Note. Fraleigh and Beauregard also give an example of a linear functional $T : \mathcal{F} \to \mathbb{R}$ defined for given $c \in \mathbb{R}$ as $T(f) = f(c)$. This is an example of an "evaluation functional" (see Example 3.4.2). In Example 3.4.5, the authors show that $T : D_\infty \to D_\infty$ defined, for $a_0, a_1, \ldots, a_n \in \mathbb{R}$, as

$$T(f) = a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x)$$

is a linear transformation. This example plays a fundamental role in the study of nth-order linear differential equations with constant coefficients (where the tools developed for matrices are useful).

Theorem 3.5. Preservation of Zero and Subtraction

Let V and V' be vectors spaces, and let $T : V \to V'$ be a linear transformation. Then

(1) $T(\vec{0}) = \vec{0}'$, and

(2) $T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2)$, for any vectors $\vec{v}_1$ and $\vec{v}_2$ in V.

Theorem 3.6. Bases and Linear Transformations.

Let $T : V \to V'$ be a linear transformation, and let B be a basis for V. For any vector $\vec{v}$ in V, the vector $T(\vec{v})$ is uniquely determined by the vectors $T(\vec{b})$ for all $\vec{b} \in B$. In other words, if two linear transformations have the same value at each basis vector $\vec{b} \in B$, then the two transformations have the same value at each vector in V.

Theorem 3.7. Preservation of Subspaces.

Let V and V' be vector spaces, and let $T : V \to V'$ be a linear transformation.

(1) If W is a subspace of V, then $T[W]$ is a subspace of V'.

(2) If W' is a subspace of V', then $T^{-1}[W']$ is a subspace of V.

Theorem 3.4.A. (Page 229 number 46) Let $T : V \to V'$ be a linear transformation and let $T(\vec{p}) = \vec{b}$ for a particular vector $\vec{p}$ in V. The solution set of $T(\vec{x}) = \vec{b}$ is the set $\{\vec{p} + \vec{h} \mid \vec{h} \in \ker(T)\}$.

Definition. A transformation $T : V \to V'$ is *one-to-one* if $T(\vec{v}_1) = T(\vec{v}_2)$ implies that $\vec{v}_1 = \vec{v}_2$ (or by the contrapositive, $\vec{v}_1 \neq \vec{v}_2$ implies $T(\vec{v}_1) \neq T(\vec{v}_2)$). Transformation T is *onto* if for all $\vec{v'} \in V'$ there is a $\vec{v} \in V$ such that $T(\vec{v}) = \vec{v'}$.

Corollary 3.4.A. One-to-One and Kernel.

A linear transformation T is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

Definition 3.10. Let V and V' be vector spaces. A linear transformation $T : V \to V'$ is *invertible* if there exists a linear transformation $T^{-1} : V' \to V$ such that $T^{-1} \circ T$ is the identity transformation on V and $T \circ T^{-1}$ is the identity transformation on V'. Such T^{-1} is called an *inverse transformation* of T.

Theorem 3.8. A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto V'. When T^{-1} exists, it is linear.

Example 3.4.C. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let a be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Determine if T is invertible. If so, find its inverse.

Note. It is at this stage that Fraleigh and Beauregard introduce the Fundamental Theorem of Finite Dimensional Vector spaces (see Theorem 3.3.A). They define an isomorphism as a one-to-one and onto linear transformation (as we did in Section 3.3, though we didn't use the language of "linear transformation" at that time). Their comments on isomorphisms on page 221 are certainly worth reading. For completeness, we now state their version of the Fundamental Theorem of Finite Dimensional Vector Spaces along with the name they give it.

Theorem 3.9. Coordinatization of Finite-Dimensional Spaces.
Let V be a finite-dimensional vector space with ordered basis $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$. The map $T : V \to \mathbb{R}^n$ defined by $T(\vec{v}) = \vec{v}_B$, the coordinate vector of $\vec{v}$ relative to B, is an isomorphism. That is, any n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Note. Just as matrices represented linear transformations mapping $\mathbb{R}^n \to \mathbb{R}^m$ (see Corollary 2.3.A, "Standard Matrix Representation of Linear Transformations"), we can use the coordinatization of general finite dimensional vector spaces V and V' to represent a linear transformation mapping $V \to V'$ with a matrix.

Theorem 3.10. Matrix Representations of Linear Transformations. Let V and V' be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for V and V', respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix A whose jth column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

Definition 3.11. The matrix A of Theorem 3.10 is the *matrix representation of T relative to B, B'.*

Examples. Page 227 Number 18, Page 227 Number 22, Page 227 Number 24.

Note. Let $T : V \to V'$ where B is a basis for V and B' is a basis for V'. By Theorem 3.8, T^{-1} is linear when it exists. So it has a matrix representation relative the B', B. The next result gives this matrix representation in terms of the matrix representation of T relative to B, B'.

Theorem 3.4.B. The matrix representation of T^{-1} relative to B', B is the inverse of the matrix representation of T relative to B, B'.

Examples. Page 228 Number 28, Page 229 Number 44, Page 226 Number 12.

Revised: 10/24/2018