Chapter 3. Vector Spaces

3.5 Inner-Product Spaces

Note. In this chapter we have seen many examples of vector spaces beyond the example $\mathbb{R}^n$ form Chapters 1 and 2. In the previous section we introduced the concept of a linear transformation between general vector spaces. Now we extend the idea of "dot product" from $\mathbb{R}^n$ to the setting of general vector spaces. This extension is called an "inner product." Similar to the application of dot products in $\mathbb{R}^n$ in Section 1.2, we use inner products on general vector spaces to define a norm on vector spaces, the distance between two vectors, and the angle between two vectors (in particular, we define what it means for two vectors to be orthogonal).

Note. Motivated by the properties of dot product on $\mathbb{R}^n$, we define the following:

Definition 3.12. An *inner product* on a vector space V is a function that associates with each ordered pair of vectors $\vec{v}, \vec{w} \in V$ a real number, written $\langle \vec{v}, \vec{w} \rangle$, satisfying the following properties for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all scalars r:

P1. Symmetry: $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$

P2. Additivity: $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$,

P3. Homogeneity: $r\langle \vec{v}, \vec{w} \rangle = \langle r\vec{v}, \vec{w} \rangle = \langle \vec{v}, r\vec{w} \rangle$,

P4. Positivity: $\langle \vec{v}, \vec{v} \rangle \geq 0$, and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

An *inner-product space* is a vector space V together with an inner product on V.

Example. Dot product on $\mathbb{R}^n$ is an example of an inner product: $\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}$.

Example. Page 236 Number 2.

Note. In fact, an inner-product space has all the geometry of $\mathbb{R}^n$! As evidence for this claim, when we define a norm and orthogonality in a general vector space we will see that the Pythagorean Theorem holds (see Exercise 23). This tells us that the geometry of an inner-product space is Euclidean (since the Pythagorean Theorem is equivalent to the Parallel Postulate from Euclidean geometry) and so is the same as the Euclidean spaces $\mathbb{R}^n$.

Example. Page 231 Example 3. Show that the space $P_{0,1}$ of all polynomial functions with real coefficients and domain $0 \leq x \leq 1$ is an inner-product space if for p and q in $P_{0,1}$ we define

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\, dx.$$

In fact, this is a special case of a more general type of inner-product space, as illustrated in the following example. (Recall from Calculus 1 that every continuous function on $a \leq x \leq b$ is Riemann integrable' some badly discontinuous functions are not Riemann integrable).

Example. Page 236 Number 10.

Definition 3.13. Let V be an inner-product space. The *magnitude* or *norm* of a vector $\vec{v} \in V$ is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. The *distance* between $\vec{v}$ and $\vec{w}$ in an inner-product space V is $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$.

Note. In $\mathbb{R}^n$ with dot product as inner product, we find that the distance "induced by this inner-product" is (as expected):

$$\begin{aligned} d(\vec{v}, \vec{w}) &= \sqrt{\vec{v} \cdot \vec{w}} \\ &= \sqrt{(v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n)} \\ &= \sqrt{v_1 w_1 + v_2 w_2 + \cdots + v_n w_n}. \end{aligned}$$

Note. Now that we have an idea of distance we can make approximation of some functions (such as continuous functions) with other functions (such as polynomials). This is an extremely useful thing to do in applied math. In fact, this is what you do in Calculus 2 when you approximate a function such as $f(x) = \cos x$ with a Taylor Polynomial (see pages 4 and 5 of my online Calculus 2 notes at: `http://faculty.etsu.edu/gardnerr/1920/12/c10s8.pdf`). Notice Fraleigh and Beauregard's quote on page 233:

> "The notion (used in Example 4) of distance between functions over an interval is very important in advanced mathematics, where it is used in approximating a complicated function over an interval as closely as possible by a function that is easier to handle."

Note. Just as in Theorem 1.2, "Properties of the Norm in $\mathbb{R}^n$," when we established that the $\mathbb{R}^n$ norm satisfies positivity, homogeneity, and the Triangle Inequality, we needed to verify that our norm $\|\cdot\|$ defined as $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ also satisfies these properties. Since the inner product satisfies positivity (by Definition 3.12(P4), "Inner-Product Space"), then $\|\cdot\|$ satisfies positivity. The argument for homogeneity is given on Page 234 Example 5. We now prove the Triangle Inequality in an inner-product space. Similar to the case of $\mathbb{R}^n$ in Section 1.2, we first prove a version of Schwarz's Inequality for an inner-product space.

Theorem 3.11. Schwarz Inequality.
Let V be an inner-product space, and let $\vec{v}, \vec{w} \in V$. Then $\langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\|\|\vec{w}\|$.

Note. We now prove the Triangle Inequality properties for an inner-product space. Since the properties of the dot product on $\mathbb{R}^n$ is pretty must the same as the properties of an inner-product on a general vector space, the proof of the Triangle Inequality her is virtually identical to the proof in $\mathbb{R}^n$ (see Theorem 1.5).

Theorem 3.5.A. The Triangle Inequality.
Let $\vec{v}, \vec{w} \in V$ (where V is an inner-product space). Then

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

Proof. We have

$$\begin{aligned}
\|\vec{v}+\vec{w}\|^2 &= \langle \vec{v}+\vec{w}, \vec{v}+\vec{w}\rangle \text{ by Definition 3.13, ``norm''}\\
&= \langle \vec{v}+\vec{w}, \vec{v}\rangle + \langle \vec{v}+\vec{w}, \vec{w}\rangle \text{ by P2}\\
&= \langle \vec{v}, \vec{v}+\vec{w}\rangle + \langle \vec{w}, \vec{v}+\vec{w}\rangle \text{ by P1}\\
&= \langle \vec{v}, \vec{w}\rangle + 2\langle \vec{v}, \vec{w}\rangle + \langle \vec{w}, \vec{w}\rangle \text{ by P1 and P2}\\
&= \|\vec{v}\|^2 + 2\langle \vec{v}, \vec{w}\rangle + \|\vec{w}\|^2 \text{ by Definition 3.13, ``norm''}\\
&\le \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2 \text{ by Schwarz Inequality, Thm. 3.11}\\
&= (\|\vec{v}\| + \|\vec{w}\|)^2
\end{aligned}$$

Taking square roots, we have the Triangle Inequality. ∎

Note. Similar to $\mathbb{R}^n$ (see Section 1.2) we can use the inner product to define the angle between two vectors. In a general vector space we are unlikely to actually compute an angle between vectors; the important property is that of orthogonality.

Definition. Let $\vec{v}, \vec{w} \in V$ where V is an inner-product space. Define the *angle between vectors* $\vec{v}$ *and* $\vec{w}$ as

$$\theta = \arccos \frac{\langle \vec{v}, \vec{w}\rangle}{\|\vec{v}\|\|\vec{w}\|}.$$

In particular, $\vec{v}$ and $\vec{w}$ are *orthogonal* (or *perpendicular*) if $\langle \vec{v}, \vec{w}\rangle = 0$.

Examples. Page 236 Number 12, Page 237 Number 18, Page 237 Number 20, Page 237 Number 24, and page 237 number 26.