Chapter 4. Determinants

4.1 Areas, Volumes, and Cross Products

Note. Determinants play a central role in matrix theory (historically, in fact, the idea of a determinant predates the idea of a matrix; see the Historical Note on page 251). In this section, we motivate the study of determinants by considering areas of parallelograms in $\mathbb{R}^2$ and volumes of "boxes" in $\mathbb{R}^3$. We define the cross product of two vectors in $\mathbb{R}^3$ and consider some of the properties.

Note. Area of a Parallelogram.

Consider the parallelogram determine by two vectors $\vec{a}$ and $\vec{b}$:

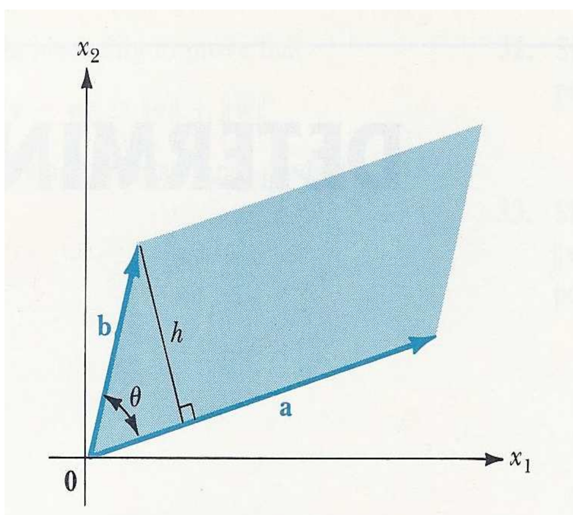

Figure 4.1, Page 239.

Its area is

$$A = \text{Area} = (\text{base}) \times (\text{height}) = \|\vec{a}\|\|\vec{b}\| \sin\theta$$
$$= \|\vec{a}\|\|\vec{b}\|\sqrt{1 - \cos^2\theta}.$$

Squaring both sides:

$$\begin{aligned} A^2 &= \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta) \\ &= \|\vec{a}\|^2\|\vec{b}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta \\ &= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2. \end{aligned}$$

Converting to components $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$ gives

$$A^2 = (a_1b_2 - a_2b_1)^2$$

or $A = |a_1b_2 - a_2b_1|$.

Definition. For a 2×2 matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, define the *determinant* of A as

$$\det(A) = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Example. Page 248 Number 20, Page 249 Number 26.

Definition. For two vectors $\vec{b} = [b_1, b_2, b_3]$ and $\vec{c} = [c_1, c_2, c_3]$ define the *cross product* of $\vec{b}$ and $\vec{c}$ as

$$\vec{b}\times\vec{c} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\hat{k}.$$

Note 4.1.A. We can take dot products and find that $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. $\vec{b} \times \vec{c}$ can be computed as:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k}.$$

Note. If $\vec{b}, \vec{c} \in \mathbb{R}^3$ are not parallel, then there are two directions perpendicular to both of these vectors. We can determine the direction of $\vec{b} \times \vec{c}$ by using a "right hand rule." If you curl the fingers of your right hand from vector $\vec{b}$ to vector $\vec{c}$, then your thumb will point in the direction of $\vec{b} \times \vec{c}$:

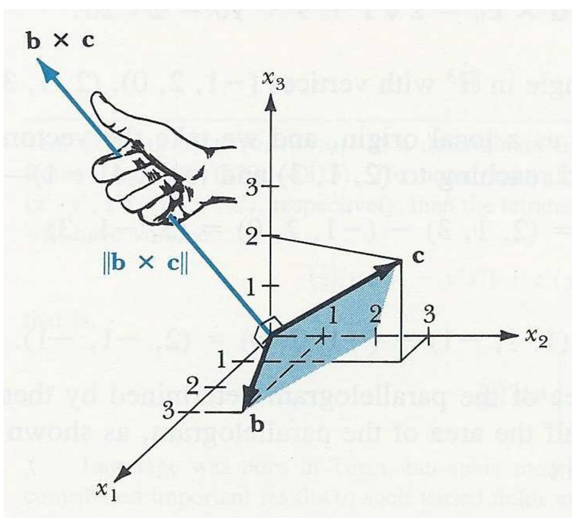

Figure 4.3, Page 242.

Example. Page 248 Number 16.

Definition. For a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ define the *determinant* as

$$\det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Note. We can now see that cross products can be computed using determinants:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Theorem 4.1.A. The area of the parallelogram determined by $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ is $\|\vec{b} \times \vec{c}\|$.

Theorem 4.1.B. The volume of a box determined by vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ is

$$V = |a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)| = |\vec{a} \cdot \vec{b} \times \vec{c}|.$$

Note. The volume of a box determined by $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ can be computed in a similar manner to cross products:

$$V = |\det(A)| = \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|.$$

Example. Page 249 Number 38, Page 249 Number 50.

Theorem 4.1. Properties of Cross Product.

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.

(1) Anticommutivity: $\vec{b} \times \vec{c} = -\vec{c} \times \vec{b}$

(2) Nonassociativity of $\times$: $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ (That is, equality does not in general hold.)

(3) Distributive Properties: $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

(4) Perpendicular Property: $\vec{b} \cdot (\vec{b} \times \vec{c}) = (\vec{b} \times \vec{c}) \cdot \vec{c} = 0$

(5) Area Property: $\|\vec{b} \times \vec{c}\|$ = Area of the parallelogram determined by $\vec{b}$ and $\vec{c}$

(6) Volume Property: $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \pm$Volume of the box determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

(7) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Note. The proof of (1) is in the supplement and is Page 247 Example 8. The proof of (3) is in the supplement and is Page 249 Number 58. The proof of (4) is mentioned in Note 4.1.A. The proofs of (2) and (6) are Exercises 57 and 59. The proof of (5) is given in Theorem 4.1.A and the proof of (6) is given in Theorem 4.1.B.

Example. Page 249 Number 56.

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