Chapter 4. Determinants

4.3 Computation of Determinants and Cramer's Rule

Note. In this section we discuss using the properties of determinants in relation to elementary row operations which were introduced the previous section. We also give an alternative technique (Cramer's Rule) for solving a system of equations which is of theoretical interest, but is too computationally complex to be of computational interest. We also give an alternative approach to finding the inverse of an invertible matrix (also of theoretical interest and not of computational interest except for very small matrices).

Note. Based on the Theorem 4.2.A, "Properties of the Determinants," we have the following technique.

Computation of A Determinant.

The determinant of an $n \times n$ matrix A can be computed as follows:

1. Reduce A to an echelon form using only row (column) addition and row (column) interchanges.

2. If any matrices appearing in the reduction contain a row (column) of zeros, then $\det(A) = 0$.

3. Otherwise (in which case the matrix is upper triangular with n pivots),

$$\det(A) = (-1)^r \cdot (\text{product of diagonal entries})$$

where r is the number of row (column) interchanges.

Example. Page 271 Number 6.

Note. We now give a way to solve a system $A\vec{x} = \vec{b}$ where A is invertible. The technique gives the unique solution in terms of determinants.

Theorem 4.5. Cramer's Rule.
Consider the linear system $A\vec{x} = \vec{b}$, where $A = [a_{ij}]$ is an $n \times n$ invertible matrix,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

The system has a unique solution given by

$$x_k = \frac{\det(B_k)}{\det(A)} \text{ for } k = 1, 2, \ldots, n,$$

where B_k is the matrix obtained from A by replacing the kth column vector of A by the column vector $\vec{b}$.

Example. Page 272 Number 26.

Note. The previous example, which requires the computation of three determinants of 2×2 matrices, hints at how inefficient Cramer's Rule is. If we use Cramer's Rule to solve a system of 3 equations in 3 unknowns (as is required in Exercises 28–31) then we must calculate four determinants of 3×3 matrices. Fraleigh and Beauregard describe a system of 10 equations in 10 unknowns and state: "This illustrates the folly of using Cramer's rule to solve linear systems." (See page 268.)

Note. We now introduce a new matrix operation which produces a new $n \times n$ matrix based on a given $n \times n$ matrix A. This new matrix is related to A^{-1}.

Note. Recall that a'_{ij} is plus or minus the determinant of the minor matrix associated with element a_{ij} (i.e., the *cofactor* of a_{ij} in A).

Definition. For an $n \times n$ matrix $A = [a_{ij}]$, define the *adjoint* of A as

$$\mathrm{adj}(A) = (A')^T$$

where $A' = [a'_{ij}]$.

Examples. Page 272 Number 18 (find the adjoint).

Theorem 4.6. Property of the Adjoint.

Let A be $n \times n$. Then

$$(\mathrm{adj}(A))A = A\,\mathrm{adj}(A) = (\det(A))\mathcal{I}.$$

Corollary 4.3.A. A Formula for A^{-1}.

Let A be $n \times n$ and suppose $\det(A) \neq 0$. Then

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A).$$

Examples. Page 272 number 18 (use the Corollary 4.3.A to find A^{-1}), Page 272 Number 22.

Note. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $\det(A) = ad - bc$, so

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Examples. Page 273 Number 36, Page 273 Number 38

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