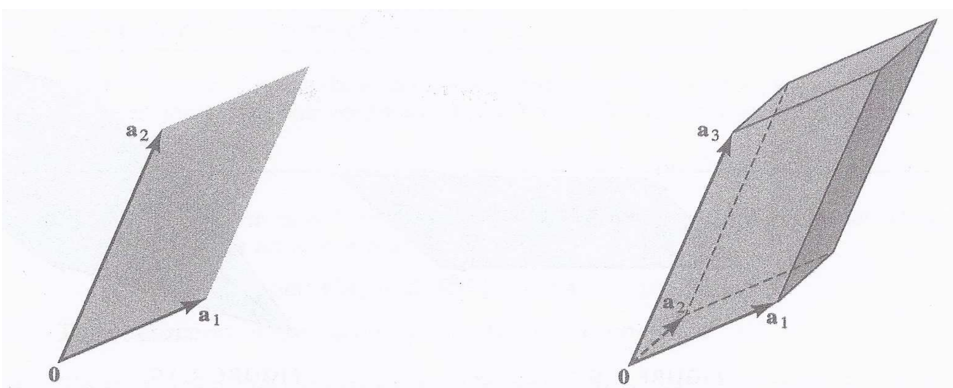
Chapter 4. Determinants

4.4 Linear Transformations and Determinants

Note. In this section we define the volume of a box in $\mathbb{R}^n$ and compute this volume using determinants (in Theorem 4.7). We then see how a linear transformation affects volumes of regions (in Theorems 4.8 and 4.9); this also involves determinants.

Note. The idea of an n-box in $\mathbb{R}^m$ is inspired by the notion of a parallelogram in $\mathbb{R}^2$ (which is determined by two vectors, say $\vec{a}_1$ and $\vec{a}_2$), and a box in $\mathbb{R}^3$ (determined by three vectors, say $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$).



Since these objects can actually be defined using two vectors or three vectors, respectively, regardless of the number of components which the vectors have, then we could just as easily discuss them as elements of $\mathbb{R}^m$ (provided $n \geq 2$ for a parallelogram and $n \geq 3$ for a box). In addition, we could consider additional "edge vectors" to make boxes of larger dimensions.

Definition 4.2. An n-Box in $\mathbb{R}^m$.

Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ be n-independent vectors in $\mathbb{R}^m$ for $n \le m$. The *n-box* in $\mathbb{R}^m$ determined by these vectors is the set of all vectors $\vec{x}$ satisfying

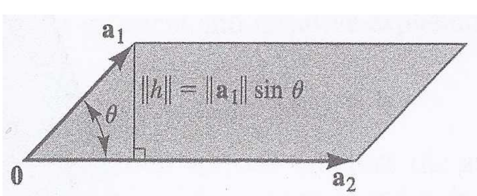
$$\vec{x} = t_1\vec{a}_1 + t_2\vec{a}_2 + \cdots + t_n\vec{a}_n$$

for $0 \le t_i \le 1$ and $i = 1, 2, \ldots, n$.

Note. We may also consider the case where $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are not independent; in this case the vectors determine a *degenerate* n-box; we'll see that the volume of such an n-box is 0.

Note. We wish to define the volume of an n-box. We do so in such a way that the volume of a 1-box is its length, then volume of a 2-box is its area, and the volume of a 3-box is its volume. It might be more appropriate to use the term "n-volume" since we do not keep up with units (if we measure length in feet, then an n-volume would be measured in foot^n).

Note. For $\vec{a}_1 \in \mathbb{R}^m$, the 1-box determined by $\vec{a}_1$ has volume $\|\vec{a}_1\| = \sqrt{\vec{a}_1 \cdot \vec{a}_1} = \sqrt{\det[\vec{a}_1 \cdot \vec{a}_1]}$.



For $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^m$, the 2-box determined by $\vec{a}_1$ and $\vec{a}_2$ has volume

$$\text{volume} = (\text{height})(\text{base}) = (\|\vec{a}_1\| \sin\theta)\|\vec{a}_1\|,$$

or

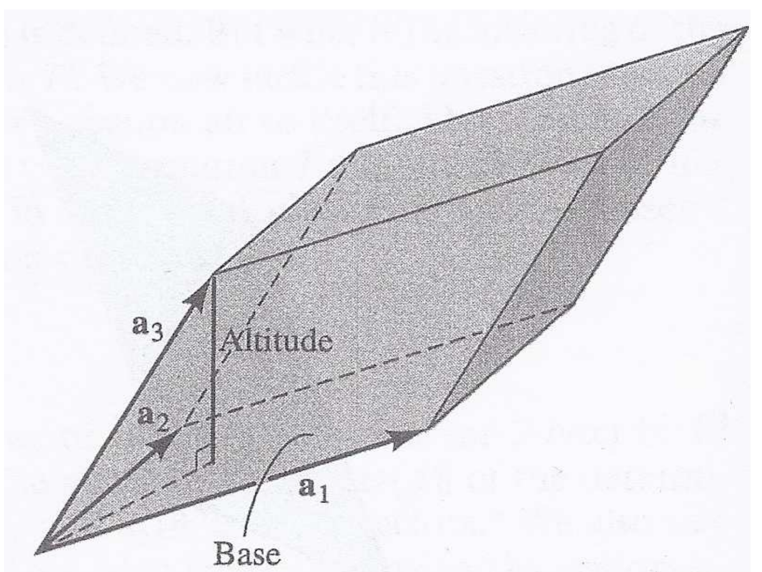
$$(\text{volume})^2 = \|\vec{a}_1\|^2\|\vec{a}_2\|^2 \sin^2\theta = \|\vec{a}_1\|^2\|\vec{a}_2\|^2(1-\cos^2\theta)$$

$$= \|\vec{a}_1\|^2\|\vec{a}_2\|^2 - (\|\vec{a}_1\|\|\vec{a}_2\|\cos\theta)^2 = (\vec{a}_1\cdot\vec{a}_1)(\vec{a}_2\cdot\vec{a}_2) - (\vec{a}_1\cdot\vec{a}_2)(\vec{a}_2\cdot\vec{a}_1)$$

$$= \begin{vmatrix} \vec{a}_1\cdot\vec{a}_1 & \vec{a}_1\cdot\vec{a}_2 \\ \vec{a}_2\cdot\vec{a}_1 & \vec{a}_2\cdot\vec{a}_2 \end{vmatrix} = |[\vec{a}_1\ \vec{a}_2]^T[\vec{a}_1\ \vec{a}_2]| = |A^TA| = \det(A^TA)$$

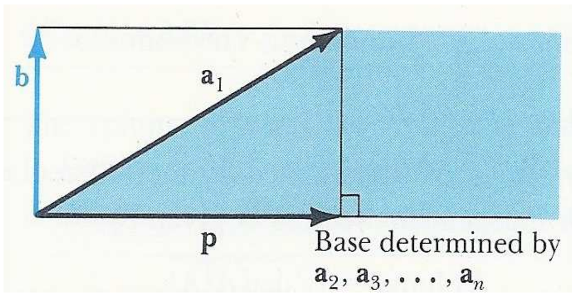
where A is the $m \times 2$ matrix with columns of $\vec{a}_1$ and $\vec{a}_2$.

Note. To motivate our recursive definition of the volume of an n-box we consider the 3-box in $\mathbb{R}^m$ determined by $\vec{a}_1, \vec{a}_2, \vec{a}_3$.



The volume of the 3-box is the volume of the 2-box determined be $\vec{a}_1$ and $\vec{a}_2$ times the altitude of the 3-box which is $\|\vec{a}_3 - \vec{p}\|$ where $\vec{p}$ is the projection of $\vec{a}_j$ onto $\text{sp}(\vec{a}_1, \vec{a}_2)$, $\vec{p} = \text{proj}_{\text{sp}(\vec{a}_1,\vec{a}_2)}(\vec{a}_3)$. However, in our recursive definition we will interchange the roles of $\vec{a}_1$ and $\vec{a}_n$.

Note. In our recursive definition, we treat the base of the n-box as the $(n-1)$-box determined by $\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ and the altitude is $\|\vec{b}\|$ where $\vec{b} = \vec{a}_1 - \vec{p} = \vec{a}_1 - \text{proj}_{\text{sp}(\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n)}(\vec{a}_1)$.



Definition B.1. Volume of an n-Box

The *volume* of the 1-box determined by nonzero vector $\vec{a}_1 \in \mathbb{R}^m$ is $\|\vec{a}_1\|$. Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ be an ordered sequence of n independent vectors, and suppose that the volume of an r-box determined by an ordered sequence of r independent vectors has been defined for $r < n$. The *volume of the n-box* in $\mathbb{R}^m$ determined by the ordered sequence $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is product of the volume of the "base" determined by the ordered sequence $\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ and the length of the vector $\vec{b} = \vec{a}_1 = \text{proj}_{\text{sp}(\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n)}(\vec{a}_1)$. That is,

$$(\text{Volume}) = \|\vec{b}\|(\text{Volume of the base})$$

$$= \|\vec{b}\|(\text{Volume of the } (n-1)\text{-box determined by } \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n).$$

Note. We'll see an efficient way to calculate the volume of an n-box below. First, we need a preliminary result.

Theorem B.2. Property of det(A^TA).

Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$ and let A be the $m \times n$ matrix with jth column $\vec{a}_j$. Let B be the $m \times n$ matrix obtained from A by replacing the first column of A by the vector $\vec{b} = \vec{a}_1 - r_2\vec{a}_2 - r_3\vec{a}_3 - \cdots - r_n\vec{a}_n$ for scalars $r_2, r_3, \ldots, r_n$. Then $\det(A^TA) = \det(B^TB)$.

Theorem 4.7. Volume of an n-Box.

The volume of the n-box in $\mathbb{R}^m$ determined by independent vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is given by (Volume) $= \sqrt{\det(A^TA)}$ where A is the $m \times n$ matrix with $\vec{a}_j$ as its jth column vector.

Example. Page 284 Number 4.

Corollary. Independence of Order.

The volume of a box determined by the independent vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ (as defined in Definition B.1) is independent of the order of the vectors.

Note. Our initial motivations for considering determinants was related to areas and volumes. In Section 4.2, we saw that the area of the parallelogram in $\mathbb{R}^2$ determined by vectors $\vec{a}_1$ and $\vec{a}_2$ is $|\det(A)|$ where A has $\vec{a}_1$ and $\vec{a}_2$ as its columns, and the volume of the box in $\mathbb{R}^3$ determined by vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is $|\det(A)|$ where A has $\vec{a}_1, \vec{a}_2, \vec{a}_3$ as it columns. (We originally had the *rows* of A as the given vectors, but from Theorem 4.2.A(1) we know $\det(A) = \det(A^T)$ so that the claims made here hold.) The next result shows that this pattern holds in n-dimensions.

Corollary. Volume of an n-Box in $\mathbb{R}^n$.

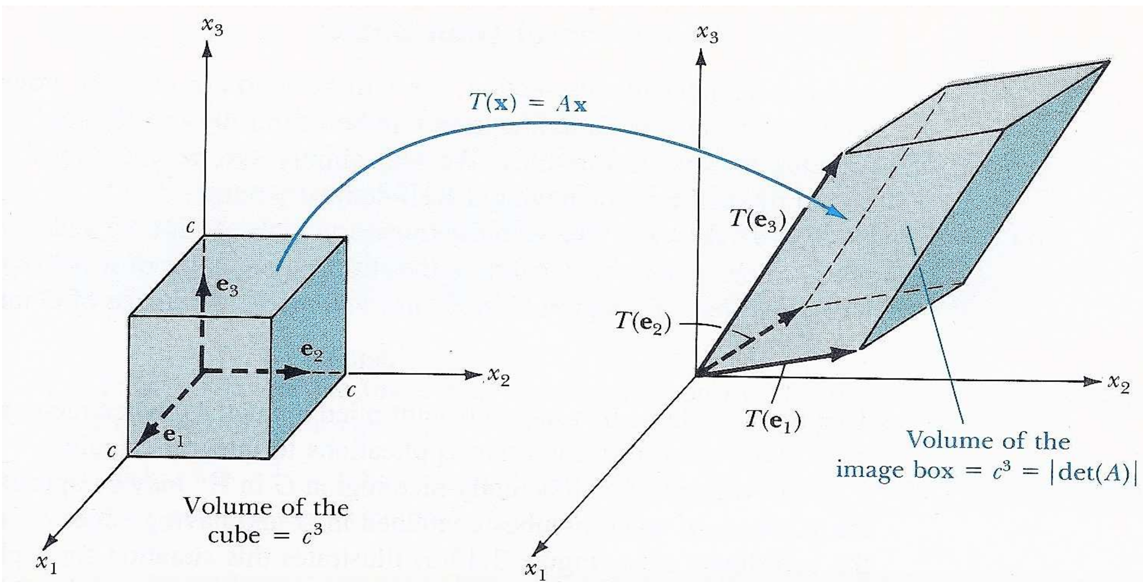
If A is an $n \times n$ matrix with independent column vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ then $|\det(A)|$ is the volume of the n-box in $\mathbb{R}^n$ determined by these n vectors.

Example. Page 284 Number 8.

Note. We now consider how a linear transformation affects the volume of an n-box in $\mathbb{R}^n$. Let $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \in \mathbb{R}^n$ determine an n-box in $\mathbb{R}^n$ of volume $|\det(B)| \neq 0$ (where the jth column of B is $\vec{b}_j$; by the second Corollary to Theorem 4.7). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix representation A. Then T maps $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \in \mathbb{R}^n$ to $A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_n \in \mathbb{R}^n$ which determine a new box in $\mathbb{R}^n$ of volume $|\det(AB)|$. By Theorem 4.4, "The Multiplicative Property," this volume is $|\det(AB)| = |\det(A)||\det(B)|$; so the volume of the new box (that is, the image of the original box under T) is $|\det(A)|$ times the volume of the original box.

Definition. For $T : \mathbb{R}^n \to \mathbb{R}^n$ an invertible linear transformation, the *volume-change factor* is $|\det(A)|$ where A is the standard matrix representation of T.

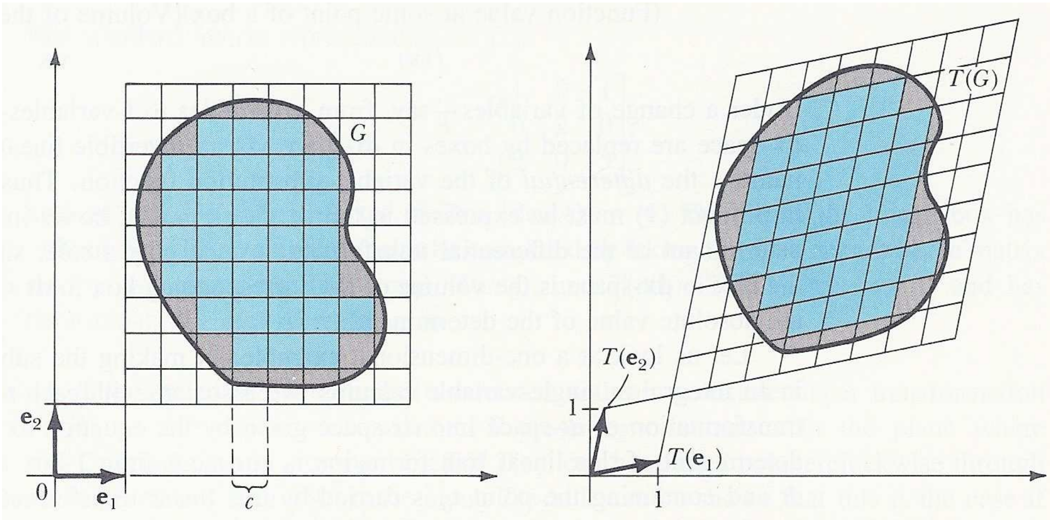
Note. If we consider the n-cube in $\mathbb{R}^n$ determined by $c\hat{e}_1, c\hat{e}_2, \ldots, c\hat{e}_n$ where $c > 0$, then this cube has volume c^n. The figure illustrates the image of such a cube in the case $n = 3$ and its image under a transformation T. Notice that the image of the cube in this case is not a cube nor even even a rectangular box; in this example it is a "skew box."



Example. Page 284 Number 22.

Note. We now give Fraleigh and Beauregard's informal argument that the volume-change factor is valid for any "sufficiently nice region" in $\mathbb{R}^n$, not just to n-cubes.

Note. Consider:



(a) Region G in the domain of T (b) The image of G under T

Region G is partitioned into little squares, each with sides parallel to the axes and of length c. As $c \to 0$ the sum of the areas of the colored squares inside the region G approaches the areas of the region. These squares are mapped by T into parallelograms of areas $c^2|\det(A)|$ where A is the standard matrix representation of T. As $c \to 0$ the sum of the areas of the colored parallelograms inside the region $T(G)$ approaches the area of the region $T(G)$. So the area of $T(G)$ is $|\det(A)|$ times the area of G. A similar construction is valid in $\mathbb{R}^n$ to conclude that the volume of region $T(G)$ in $\mathbb{R}^n$ is $|\det(A)|$ times the volume of regions G in $\mathbb{R}^n$. We summarize this as follows.

Theorem 4.8. Volume-Change Factor for $T : \mathbb{R}^n \to \mathbb{R}^n$.
Let G be a region in $\mathbb{R}^n$ of volume V and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of rank n with standard matrix representation A. Then the volume of the image of G under T is $|\det(A)|V$.

Note. We can similarly address volume change for a transformation from $\mathbb{R}^n \to \mathbb{R}^m$ (where $m \geq n$), as follows.

Theorem 4.9. Volume-Change Factor for $T : \mathbb{R}^n \to \mathbb{R}^m$.
Let G be a region in $\mathbb{R}^n$ of volume V. Let $m \geq n$ and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation of rank n. Let A be the standard matrix representation of T. Then the volume of the image of G in $\mathbb{R}^m$ under the transformation T is $\sqrt{\det(A^T A)}V$.

Example. Page 284 Number 32.

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