Chapter 5. Eigenvalues and Eigenvectors

5.3 Two Applications

Note. The "two applications" in the title of this section involve (1) long-term behavior of systems of the form $A^k\vec{x}$ where $\vec{x}$ is some initial vector, and (2) systems of linear differential equations with constant coefficients.

Note. We start with an initial "information vector" $\vec{x}$ and a matrix A such that at "stage" k the information matrix is $A^k\vec{x}$. A Markov chain is an example of such a process (see Section 1.7).

Note. Let $n \times n$ matrix A be diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where $B = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$; that is, let A be diagonalizable (see Corollary 1, "A Criterion for Diagonalization," of Section 5.2). If we express initial information vector $\vec{x}$ relative to ordered basis B, we get $\vec{x} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$. Then

$$\begin{aligned} A^k\vec{x} &= A^k(d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n) \\ &= d_1A^k\vec{v}_1 + d_2A^k\vec{v}_2 + \cdots + d_nA^k\vec{v}_n \\ &= d_1\lambda_1^k\vec{v}_1 + d_2\lambda_2^k\vec{v}_2 + \cdots + d_n\lambda^k\vec{v}_n \text{ by Theorem 5.1(1),} \\ &\qquad \text{"Properties of Eigenvalues and Eigenvectors."} \end{aligned}$$

If we index the eigenvalues such that $|\lambda_i| \geq |\lambda_j|$ if $i < j$ (so that $|\lambda_1|$ is a largest eigenvalue) then we have

$$\begin{aligned} A^k\vec{x} &= d_1\lambda_1^k\vec{v}_1 + d_2\lambda_2^k\vec{v}_2 + \cdots + d_n\lambda_n^k\vec{v}_n \\ &= \lambda_1^k(d_1\vec{v}_1 + d_2(\lambda_2/\lambda_1)^k\vec{v}_2 + \cdots + d_n(\lambda_n/\lambda_1)^k\vec{v}_n) \qquad (1) \end{aligned}$$

Thus if k is large and $d_1 \neq 0$ the vector $A^k\vec{x}$ is approximately equal to $d_1\lambda_1^k\vec{v}_1$ in the sense that $\|A^k\vec{x} - d_1\lambda_1^k\vec{v}_1\|$ is small compared with $\|A^k\vec{x}\|$.

Page 318 Example 2. At the beginning of Section 5.1 we described the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ where each new number is the sum of the previous two. With $F_0 = 0$, $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we can find

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^k\vec{x}.$$

Consider

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - (1)(1) = \lambda^2 - \lambda - 1 = 0.$$

So $\lambda = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$. We take $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$. If $\vec{v}_1 = [v_1, v_2]^T$ is an eigenvector corresponding to eigenvalue $\lambda_1 = (1+\sqrt{5})/2$, then we need $(A - \lambda_1\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix:

$$\left[\begin{array}{cc|c} 1-(1+\sqrt{5})/2 & 1 & 0 \\ 1 & -(1+\sqrt{5})/2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} (1-\sqrt{5})/2 & 1 & 0 \\ 1 & (-1-\sqrt{5})/2 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & (-1-\sqrt{5})/2 & 0 \\ (1-\sqrt{5})/2 & 1 & 0 \end{array}\right] \underset{\sim}{R_2 \to R_2 - ((1-\sqrt{5})/2)R_1} \left[\begin{array}{cc|c} 1 & (-1-\sqrt{5})/2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so we need $\begin{array}{rcl} v_1 - ((1+\sqrt{5})/2)v_2 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} v_1 & = & ((1+\sqrt{5})/2)v_2 \\ v_2 & = & v_2 \end{array}$ or, with $r = v_2$ as a free variable, $\begin{array}{rcl} v_1 & = & ((1+\sqrt{5})/2)r \\ v_2 & = & r \end{array}$. We choose $r = 2$ to get the eigenvector $\vec{v}_1 = [1+\sqrt{5}, 2]^T$. If $\vec{v}_2 = [v_1, v_2]^T$ is an eigenvalue corresponding to eigenvalue $\lambda_2 = (1-\sqrt{5})/2$ then we need $(A - \lambda_2 \mathcal{I})\vec{v}_2 = \vec{0}$. So we consider the augmented matrix

$$\left[\begin{array}{cc|c} 1-(1-\sqrt{5})/2 & 1 & 0 \\ 1 & -(1-\sqrt{5})/2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} (1+\sqrt{5})/2 & 1 & 0 \\ 1 & (-1+\sqrt{5})/2 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & (-1+\sqrt{5})/2 & 0 \\ (1+\sqrt{5})/2 & 1 & 0 \end{array}\right] \underset{\sim}{R_2 \to R_2 - ((1+\sqrt{5})/2)R_1} \left[\begin{array}{cc|c} 1 & (-1+\sqrt{5})/2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

So we need $\begin{array}{rcl} v_1 + ((-1+\sqrt{5})/2)v_2 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} v_1 & = & ((1-\sqrt{5})/2)v_2 \\ v_2 & = & v_2 \end{array}$ or, with $s = v_2$ as a free variable, $\begin{array}{rcl} v_1 & = & ((1-\sqrt{5})/2)s \\ v_2 & = & s \end{array}$. We choose $s = 2$ to get the eigenvalue $\vec{v}_2 = [1-\sqrt{5}, 2]^T$. We define $= \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}$. To find the coordinate vector $\vec{d}$ of $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ relative to the ordered basis $B = (\vec{v}_1, \vec{v}_2)$, we want $\vec{x} = C\vec{d}$ or

$\vec{d} = C^{-1}\vec{x}$. To find C^{-1}, consider

$$[C \mid \mathcal{I}] = \left[\begin{array}{cc|cc} 1+\sqrt{5} & 1-\sqrt{5} & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 2 & 2 & 0 & 1 \\ 1+\sqrt{5} & 1-\sqrt{5} & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1 \to R_1/2} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1/2 \\ 1+\sqrt{5} & 1-\sqrt{5} & 1 & 0 \end{array}\right] \underset{\sim}{R_2 \to R_2-(1+\sqrt{5})R_1} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1/2 \\ 0 & -2\sqrt{5} & 1 & (-1-\sqrt{5})/2 \end{array}\right]$$

$$\underset{\sim}{R_2 \to R_2/(-2\sqrt{5})} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1/2 \\ 0 & 1 & -1/(2\sqrt{5}) & (1+\sqrt{5})/(4\sqrt{5}) \end{array}\right]$$

$$\underset{\sim}{R_1 \to R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/(2\sqrt{5}) & (-1+\sqrt{5})/(4\sqrt{5}) \\ 0 & 1 & -1/(2\sqrt{5}) & (1+\sqrt{5})/(4\sqrt{5}) \end{array}\right].$$

So

$$C^{-1} = \begin{bmatrix} 1/(2\sqrt{5}) & (-1+\sqrt{5})/(4\sqrt{5}) \\ -1/(2\sqrt{5}) & (1+\sqrt{5})/(4\sqrt{5}) \end{bmatrix} = \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix}.$$

Hence

$$\vec{d} = C^{-1}\vec{x} = \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

From equation (1) above, we have

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^k\vec{x} = d_1\lambda_1^k\vec{v}_1 + d_2\lambda_2^k\vec{v}_2$$

$$= \left(\frac{1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k \begin{bmatrix} 1+\sqrt{5} \\ 2 \end{bmatrix} + \left(\frac{-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k \begin{bmatrix} 1-\sqrt{5} \\ 2 \end{bmatrix}.$$

So $\boxed{F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right).}$ Notice that this is the same formula for F_k as given by Fraleigh and Beauregard even though we have used different eigenvectors.

□

Note. As can be seen in equation (1), if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are less than 1 in magnitude (absolute value for real eigenvalues and "modulus" for complex numbers) then as $k \to \infty$ we have $\lambda_i^j \to 0$ and so $A^k\vec{x} \to \vec{0}$. If some λ_i is greater than 1 in magnitude then as $k \to \infty$ we have $|\lambda_i^k| \to \infty$ and so $\|A^k\vec{x}\| \to \infty$. These observations motivate the following definition.

Definition. Let A be an $n \times n$ diagonalizable matrix with eigenvectors $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where $B = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$. Then the process $A^k\vec{x}$ is *unstable* if some $|\lambda_i| > 1$, *stable* if all $|\lambda_i| < 1$, and *neutrally stable* if all $|\lambda_i| \leq 1$ and some $|\lambda_j| = 1$. The eigenvectors of A are the *normal modes* of the process.

Note. In the process yielding the Fibonacci series we have the eigenvalue $\lambda_1 = (1+\sqrt{5})/2$ and so $|\lambda_1| > 1$. So this process is unstable. This is not surprising since the Fibonacci number of course diverges to ∞. In a Markov chain $T\vec{x}$, $T^2\vec{x}$, $T^3\vec{x}$, $\ldots$, the sum of the entries of the column vector $T^k\vec{x}$ s a constant so a Markov chain is neutrally stable.

Example. Page 325 Number 4.

Note. We saw in Example 8 of Section 5.1 (see the class notes) that the general solution to the differential equation $y' = \lambda y$ is $y = ke^{\lambda x}$ (where y is a function of x and the prime represents differentiation with respect to x). We can change variables and consider x as a function of t (think "time"), giving the differential equation $x' = ax$ with general solution $x(t) = ke^{at}$. We now use this fact to solve a system of n linear differential equations with constant coefficients.

Definition. Suppose $x_1, x_2, \ldots, x_n$ are each differentiable functions of t and that the derivatives with respect to t of these functions satisfy:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \;&\vdots\; \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

This is a system of n *linear differential equations with constant coefficients.*

Note. If we let $\vec{x} = [x_1, x_2, \ldots, x_n]^T$ be a vector of differentiable functions of t, then we can represent the above system of equations as $\vec{x}' = A\vec{x}$ where $A = [a_{ij}]$ and $\vec{x}' = [x_1', x_2', \ldots, \vec{x}_n']^T$. If A is diagonal then the system is

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}x_1 \\ a_{22}x_2 \\ \vdots \\ a_{nn}x_n \end{bmatrix}$$

and so we can solve the system one equation at a time to get

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 e^{a_{11}t} \\ k_2 e^{a_{22}t} \\ \vdots \\ k_n e^{a_{nn}t} \end{bmatrix}.$$

Note. In the event that matrix A in the system above is diagonalizable, $D = C^{-1}AC$ (where the diagonal entries of D are eigenvalues of A and the columns of C are corresponding eigenvectors of A, as in Theorem 5.2, "Matrix Summary of Eigenvalues of A") then the system $\vec{x}' = A\vec{x} = C^{-1}AC\vec{x}$ ecomes $C^{-1}\vec{x}' = D(C^{-1}\vec{x})$ or $\vec{y}' - D\vec{y}$ wher $\vec{y} = C^{-1}\vec{x}$. Then the general solution of $\vec{y}' = D\vec{y}$ is

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2 t} \\ \vdots \\ k_n e^{\lambda_n t} \end{bmatrix}$$

and then we find $\vec{x}$ from the equation $\vec{x} = C\vec{y}$. We now illustrate this process with an example.

Example. Page 325 Number 10.

Revised: 4/15/2019