Chapter 6. Orthogonality

6.1 Projections

Note. In Chapter 3, "Vector Spaces," we considered several different bases for a given finite dimensional vector space. Computationally, the easiest type of basis to use is one for which the vectors are pairwise orthogonal unit vectors. In this section we define the idea of a projection of one vector onto another and the projection of one vector onto a subspace. In Section 6.2, "The Gram-Schmidt Process," we use projections to change a given basis into a "nice" basis of orthogonal unit vectors.

Note. We want to find the projection $\vec{p}$ of vector $\vec{F}$ on $\text{sp}(\vec{a})$:

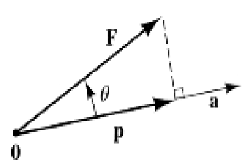


Figure 6.1, Page 327.

We see that $\vec{p}$ is a multiple of $\vec{a}$. Now $(1/\|\vec{a}\|)\vec{a}$ is a unit vector having the same direction as $\vec{a}$, so $\vec{p}$ is a scalar multiple of this unit vector. We need only find the appropriate scalar. From the above figure, we see that the appropriate scalar is $\|\vec{F}\|\cos\theta$, because it is the length of the leg labeled $\vec{p}$ of the right triangle. If $\vec{p}$ is in the opposite direction of $\vec{a}$ and $\theta \in [\pi/2, \pi]$:

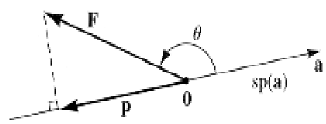


Figure 6.2, Page 327.

then the appropriate scalar is again given by $\|\vec{F}\| \cos\theta$. Thus

$$\vec{p} = \frac{\|\vec{F}\| \cos\theta}{\|\vec{a}\|}\vec{a} = \frac{\|\vec{F}\|\|\vec{a}\| \cos\theta}{\|\vec{a}\|\|\vec{a}\|}\vec{a} = \frac{\vec{F}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a}.$$

We use this to motivate the following definition.

Definition. Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ The *projection* $\vec{p}$ *of* $\vec{b}$ *on sp*$(\vec{a})$ is

$$\vec{p} = \text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a}.$$

Example. Page 336 number 4.

Note. Next, we want to project a vector onto a subspace. To do so, we first decompose the whole vector space using the given subspace.

Definition 6.1. Let W be a subspace of $\mathbb{R}^n$. The set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is the *orthogonal complement* of W and is denoted by $W^\perp$ (and read "W perp").

Note 6.1.A. To find $W^\perp$, we must find a basis for subspace W (or at least a spanning set for W). We use our knowledge of matrices and nullspaces as follows. To find the orthogonal complement of a subspace of $\mathbb{R}^n$:

1. Find a matrix A having as *row* vectors a generating set for W.

2. Find the nullspace of A — that is, the solution space of $A\vec{x} = \vec{0}$. This nullspace is $W^\perp$.

Example. Page 336 number 10.

Theorem 6.1. Properties of $W^\perp$.
The orthogonal complement $W^\perp$ of a subspace W of $\mathbb{R}^n$ has the following properties:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.

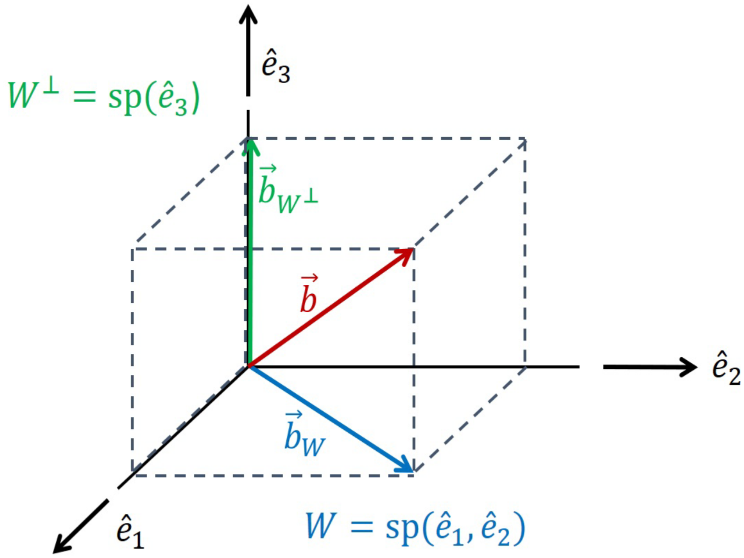
2. $\dim(W^\perp) = n - \dim(W)$.

3. $(W^\perp)^\perp = W$.

4. Each vector $\vec{b} \in \mathbb{R}^n$ can be expressed uniquely in the form $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ for $\vec{b}_W \in W$ and $\vec{b}_{W^\perp} \in W^\perp$.

Note. Theorem 6.1 (in particular, Property 4) allows us to define the projection of a vector onto a subspace of $\mathbb{R}^n$.

Definition 6.2. Let $\vec{b} \in \mathbb{R}^n$, and let W be a subspace of $\mathbb{R}^n$. Let $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ be as described in Theorem 6.1. Then $\vec{b}_W$ is the *projection of* $\vec{b}$ *on* W, denoted $\text{proj}_W(\vec{b})$.



Note 6.1.B. To find the projection of $\vec{b}$ on W, follow these steps:

1. Select a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ for the subspace W.

2. Find a basis $\{\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\}$ for $W^\perp$.

3. Find the coordinate vector $\vec{r} = [r_1, r_2, \ldots, r_n]$ of $\vec{b}$ relative to the ordered basis $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ so that

$$\vec{b} = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n.$$

4. Then $\vec{b}_W = \text{proj}_W(\vec{b}) = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k$.

Example. Page 336 number 20b.

Note. We can perform projections in inner product spaces by replacing the dot products in the formulas above with inner products.

Example. Page 335 Example 6. Consider the inner product space $\mathcal{P}_{[0,1]}$ of all polynomial functions defined on the interval $[0, 1]$ with inner product

$$\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\, dx.$$

Find the projection of $f(x) = x$ on $\mathrm{sp}(1)$ and then find the projection of x on $\mathrm{sp}(1)^{\perp}$.

Example. Page 337 number 26 and 28.

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