Chapter 6. Orthogonality

6.2 The Gram Schmidt Process

Note. In Section 6.1 we expressed a fondness (for computational simplicity) for bases of vector spaces made of pairwise orthogonal unit vectors. In this section we give a process (the Gram-Schmidt Process) which allows us to convert a given basis into a "nice" basis. The process can be computationally lengthy and is heavily based on the projections introduced in Section 6.1.

Definition. A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ of nonzero vectors in $\mathbb{R}^n$ is *orthogonal* if the vectors $\vec{v_j}$ are mutually perpendicular; that is, if $\vec{v_i} \cdot \vec{v_j} = 0$ for $i \neq j$.

Theorem 6.2. Orthogonal Bases.
Let $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be an orthogonal set of nonzero vectors in $\mathbb{R}^n$. Then this set is independent and consequently is a basis for the subspace $\text{sp}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k})$.

Note. To find the projection of vector $\vec{b}$ on to subspace W in Section 6.1 we were required to find a coordinate vector relative to a certain ordered basis. This can be simplified if we have an orthogonal basis for subspace W, as given in the following theorem.

Theorem 6.3. Projection Using an Orthogonal Basis.

Let $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, and let $\vec{b} \in \mathbb{R}^n$. The projection of $\vec{b}$ on W is

$$\begin{aligned}\vec{b}_W = \mathrm{proj}_W(\vec{b}) &= \frac{\vec{b} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}}\vec{v_1} + \frac{\vec{b} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}}\vec{v_2} + \cdots + \frac{\vec{b} \cdot \vec{v_k}}{\vec{v_k} \cdot \vec{v_k}}\vec{v_k} \\ &= \mathrm{proj}_{\vec{v}_1}(\vec{b}) + \mathrm{proj}_{\vec{v}_2}(\vec{b}) + \cdots + \mathrm{proj}_{\vec{v}_k}(\vec{b}).\end{aligned}$$

Example. Page 347 Number 4.

Definition 6.3. Let W be a subspace of $\mathbb{R}^n$. A basis $\{\vec{q_1}, \vec{q_2}, \ldots, \vec{q_k}\}$ for W is *orthonormal* if

1. $\vec{q_i} \cdot \vec{q_j} = 0$ for $i \neq j$, and

2. $\vec{q_i} \cdot \vec{q_i} = 1$.

That is, each vector of the basis is a unit vector and the vectors are pairwise orthogonal.

Note. If $\{\vec{q_1}, \vec{q_2}, \ldots, \vec{q_k}\}$ is an orthonormal basis for W, then

$$\vec{b}_W = \mathrm{proj}_W((\vec{b}) = (\vec{b} \cdot \vec{q_1})\vec{q_1} + (\vec{b} \cdot \vec{q_2})\vec{q_2} + \cdots + (\vec{b} \cdot \vec{q_k})\vec{q_k}.$$

Note. The previous note shows why it is computationally desirable to have an orthonormal basis. Notice that it only requires the computation of some dot products; recall that if we are given an arbitrary basis $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$ then to write $\vec{b}$ as a linear combination of these basis elements we must solve the system of equations $A\vec{x} = \vec{b}$ where A is a matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ (see Note 3.3.A, for example).

Theorem 6.4. Orthonormal Basis (Gram-Schmidt) Theorem.

Let W be a subspace of $\mathbb{R}^n$, let $\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_k}\}$ be any basis for W, and let

$$W_j = \text{sp}(\vec{a_1}, \vec{a_2}, \ldots, \vec{a_j}) \text{ for } j = 1, 2, \ldots, k.$$

Then there is an orthonormal basis $\{\vec{q_1}, \vec{q_2}, \ldots, \vec{q_k}\}$ for W such that $W_j = \text{sp}(\vec{q_1}, \vec{q_2}, \ldots, \vec{q_j})$.

Note. The proof of Theorem 6.4 is computational. We summarize the proof in the following procedure:

Gram-Schmidt Process.

To find an orthonormal basis for a subspace W of $\mathbb{R}^n$:

1. Find a basis $\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_k}\}$ for W.

2. Let $\vec{v_1} = \vec{a_1}$. For $j = 1, 2, \ldots, k$, compute in succession the vector $\vec{v_j}$ given by subtracting from $\vec{a_j}$ its projection on the subspace generated by its predecessors.

3. The $\vec{v_j}$ so obtained form an orthogonal basis for W, and they may be normalized to yield an orthonormal basis.

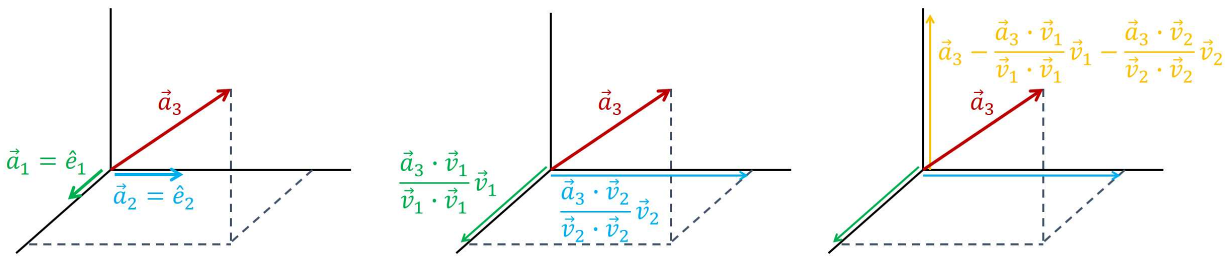
Note. We can recursively describe the way to find $\vec{v_j}$ as:

$$\vec{v_j} = \vec{a_j} - \left(\frac{\vec{a_j} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} + \frac{\vec{a_j} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} + \cdots + \frac{\vec{a_j} \cdot \vec{v}_{j-1}}{\vec{v}_{j-1} \cdot \vec{v}_{j-1}} \vec{v}_{j-1} \right) = \vec{a}_j - \text{proj}_{\text{sp}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1})}(\vec{a}_j).$$

If we normalize the $\vec{v_j}$ as we go by letting $\vec{q_j} = (1/\|\vec{v_j}\|)\vec{v_j}$, then we have

$$\vec{v_j} = \vec{a_j} - ((\vec{a_j} \cdot \vec{q_1})\vec{q_1} + (\vec{a_j} \cdot \vec{q_2})\vec{q_2} + \cdots + (\vec{a_j} \cdot \vec{q}_{j-1})\vec{q}_{j-1}).$$

Note. We can geometrically illustrate the Gram-Schmidt Process in this notation in a special case as follows:



Example. Page 348 Number 10.

Note. A matrix "factorization" involves writing a given matrix A as a product $A = BC$ where matrices B and C have some desired property. The desired properties are usually related to simplifying computations. Based on the proof of Theorem 6.4, "Orthonormal Basis (Gram-Schmidt) Theorem." we can deduce the following.

Corollary 1. ***QR*-Factorization.**
Let A be an $n \times k$ matrix with independent column vectors in $\mathbb{R}^n$. There exists an $n \times k$ matrix Q with orthonormal column vectors and an upper-triangular invertible $k \times k$ matrix R such that $A = QR$.

Example. Page 348 Number 26.

Note. We saw in Theorem 2.3(2), "Existence and Determination of Basis," that every independent set of vectors in $\mathbb{R}^n$ can be enlarged to a basis for $\mathbb{R}^n$. The next corollary to Theorem 6.4 shows that a similar result holds for every set of orthogonal vectors in $\mathbb{R}^n$.

Corollary 2. Expansion of an Orthogonal Set to an Orthogonal Basis. Every orthogonal set of vectors in a subspace W of $\mathbb{R}^n$ can be expanded if necessary to an orthogonal basis of W.

Examples. Page 348 Number 20, Page 340 Number 30, Page 340 Number 32, and Page 349 Number 34.

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