Chapter 6. Orthogonality

6.3 Orthogonal Matrices

Note. In Exercise 6.2.29 it is to be shown that an $n \times k$ matrix A has columns which are orthonormal if and only if $A^T A = \mathcal{I}$. In this section we restrict our attention to square matrix with this property. We'll see that such matrices when treated as linear transformations, preserve lengths (i.e., norms) and angles (i.e., dot products).

Definition 6.4. An $n \times n$ matrix A is *orthogonal* if $A^T A = \mathcal{I}$.

Note. Since the columns of an orthogonal matrix from an orthonormal set (as observed above), then it might seem reasonable to call them "orthonormal" matrices, but the term "orthogonal;; is standard for such matrices. Notice that in Definition 6.4 we could replace the condition $A^T A = \mathcal{I}$ with the condition $A^{-1} = A^T$ (notice that, since A is square, $AA^T = \mathcal{I}$ by Theorem 1.11, "A Commutative Property").

Theorem 6.5. Characterizing Properties of an Orthogonal Matrix.
Let A be an $n \times n$ matrix. The following conditions are equivalent:

1. The rows of A form an orthonormal basis for $\mathbb{R}^n$.

2. The columns of A form an orthonormal basis for $\mathbb{R}^n$.

3. The matrix A is orthogonal—that is, A is invertible and $A^{-1} = A^T$.

Example. Page 358 Number 2.

Note. We now show that an orthogonal matrix (when treated as a linear transformation) preserves dot products, lengths, and angles making them "especially desirable" as Fraleigh and Beauregard say (page 351).

Theorem 6.6. Properties of $A\vec{x}$ for an Orthogonal Matrix A.
Let A be an orthogonal $n \times n$ matrix and let $\vec{x}$ and $\vec{y}$ be any column vectors in $\mathbb{R}^n$. Then

1. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$,

2. $\|A\vec{x}\| = \|\vec{x}\|$, and

3. The angle between nonzero vectors $\vec{x}$ and $\vec{y}$ equals the angle between $A\vec{x}$ and $A\vec{y}$.

Example. Page 358 Number 12.

Note. In Theorem 5.5, "Diagonalization of Real Symmetric Matrices," we claimed that every real symmetric matrix is real diagonalizable (though the proof was postponed to Chapter 9 since it requires complex numbers). It turns out that we can go further and show that every real symmetric matrix A is real diagonalizable as $C^{-1}AC = D$ where D is diagonal and C (and C^{-1}) are orthogonal; in addition, all entries of C, C^{-1}, and D are real. To prove thie, we need to explore a property of the eigenvectors of real symmetric matrix A.

Theorem 6.7. Orthogonality of Eigenspaces of a Real Symmetric Matrix. Eigenvectors of a real symmetric matrix that correspond to different eigenvalues are orthogonal. That is, the eigenspaces of a real symmetric matrix are orthogonal.

Note. We now have the equipment to consider the diagonalization of real symmetric matrices by using orthogonal matrix C.

Theorem 6.8. Fundamental Theorem of Real Symmetric Matrices. Every real symmetric matrix A is diagonalizable. The diagonalization $C^{-1}AC = D$ can be achieved by using a real orthogonal matrix C.

Note. The converse of Theorem 6.8 is also true. If $D = C^{-1}AC$ is a diagonal matrix and C is an orthogonal matrix, then A is symmetric (see Page 359 Number 24). The equation $D = C^{-1}AC$ is called the *orthogonal diagonalization* of A.

Example. Page 358 Number 16.

Note. Recall that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ corresponds to some $m \times n$ matrix A (called the standard matrix representation of T) and conversely every $m \times n$ matrix A corresponds to a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ (defined as $T(\vec{x}) = A\vec{x}$). See Section 2.3, "Linear Transformations of Euclidean Spaces." So we can extend the idea of an orthogonal matrix to an orthogonal transformation. The following definition is inspired by Theorem 6.6(1) (the Preservation of Dot Product).

Definition 6.5. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if it satisfies $T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.

Note. We nwo see that the pervious definition yields the desired relationship between orthogonal matrices and orthogonal linear transformations.

Theorem 6.9. Orthogonal Transformations vis-à-vis Matrices.
A linear transformation T of $\mathbb{R}^n$ into itself is orthogonal if and only if its standard matrix representation A is an orthogonal matrix.

Example. Page 359 Number 34.

Examples. Page 359 Number 20, Page 359 Number 24, Page 359 Number 30.

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