Chapter 7. Change of Basis

7.2 Matrix Representations and Similarity

Note. The purpose of this section is to consider the effect that choosing different bases for coordinatization has on the matrix representation of a linear transformation. All results are stated in terms of $\mathbb{R}^n$ but (similar to the examples in the notes for the previous section) can be translated into applications in any finite-dimensional vector space using the Fundamental Theorem of Finite Dimensional Vector Space (the proof of which is based on coordinatization).

Note. In Theorem 3.10, "Matrix Representations of Linear Transformations," in Section 3.4 we saw that if V and V' are vector spaces with ordered bases $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$, respectively, then the matrix representation of T relative to B, B' is

$$R_{B,B'} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \cdots & T(\vec{b}_n)_{B'} \\ \vdots & \vdots & & \vdots \end{bmatrix}.$$

This matrix relates to the transformation T and coordinate vectors $T(\vec{v})_{B'} = R_{B,B'}\vec{v}_B$ for all $\vec{v} \in V$. Just as we represent compositions of linear transformations with products of their standard matrix representation in Section 2.3, if we have vector spaces V, V', and V'' with ordered bases B, B', and B'', respectively, and linear transformations $T : V \to V'$ and $T' : V' \to V''$ with matrix representations $R_{B,B'}$ and $R_{B',B''}$, respectively, then the matrix representation of transformation $T' \circ T : V \to V''$ is $R_{B,B''} = R_{B',B''}R_{B,B'}$. Here, we have for $\vec{v} \in V$,

$$((T' \circ T)\vec{v})_{B''} = (T'(T\vec{v}))_{B''} = (T'(R_{B,B'}\vec{v}_B))_{B''} = R_{B',B''}R_{B,B'}\vec{v}_B.$$

Note. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then we can find a matrix representation of it similar to what we described above by considering

$$R_B = \begin{bmatrix} \vdots & \vdots & & \vdots \\ T(\vec{b}_1)_B & T(\vec{b}_2)_B & \cdots & T(\vec{b}_n)_B \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

where $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$. If we wich to find the matrix representation of T with respect to ordered basis B', we can take $\vec{v}_{B'} \in \mathbb{R}^n$ and (1) convert it to B coordinates using $C_{B',B}$, (2) applying T using R_B, and (3) converting back to B' coordinates. This gives the relationship $R_{B'} = C_{B,B'} R_B C_{B',B}$. Since $C_{B,B'}$ and $C_{B',B}$ are inverses, we have $R_{B'}$ and R_B related as $R_{B'} = C^{-1} R_B C$ (where $C = C_{B',B}$). So we see that $R_{B'}$ and R_B are similar matrices (see Definition 5.4 in section 5.2). That is, matrix representations of the same linear transformations of the same linear transformation relative to different bases are similar. We summarize this in the following.

Theorem 7.1. Similarity of Matrix Representations of T.
Let T be a linear transformation of a finite-dimensional vector space V into itself, and let B and B' be ordered bases of V. Let R_B and $R_{B'}$ be the matrix representations of T relative to B and B', respectively. Then

$$R_{B'} = C^{-1} R_B C$$

where $C = C_{B',B}$ is the change-of-coordinates matrix from B' to B. Hence, $R_{B'}$ and R_B are similar matrices.

Example. Page 406 number 2.

Theorem 7.A. Significance of the Similarity Relationship for Matrices. Two $n \times n$ matrices are similar if and only if they are matrix representations of the same linear transformation T relative to suitable ordered bases.

Note. Certain properties of matrices are independent of the coordinate system in which they are expressed. These properties are called *coordinate-independent*. For example, we will see that the eigenvalues of a matrix are coordinate-independent quantities.

Theorem 7.2. Eigenvalues and Eigenvectors of Similar Matrices. Let A and R be similar $n \times n$ matrices, so that $R = C^{-1}AC$ for some invertible $n \times n$ matrix C. Let the eigenvalues of A be the (not necessarily distinct) numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

1. The eigenvalues of R are also $\lambda_1, \lambda_2, \ldots, \lambda_n$.

2. The algebraic and geometric multiplicity of each λ_i as an eigenvalue of A remains the same as when it is viewed as an eigenvalue of R.

3. If $\vec{v}_i \in \mathbb{R}^n$ is an eigenvector of the matrix A corresponding to λ_i, then $C^{-1}\vec{v}_i$ is an eigenvector of the matrix R corresponding to λ_i.

Note. We now give a proof of Theorem 7.2(1). Proofs of parts (2) and (3) are to be given in Page 407 Numbers 24 and 25, respectively.

Definition. The *geometric multiplicity* of an eigenvalue λ of a transformation T is the dimension of the eigenspace $E_\lambda = \{\vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v}\}$. The *algebraic multiplicity* λ is the algebraic multiplicity of the λ as a root of the characteristic polynomial of T (technically, the characteristic polynomial of the matrix which represents T).

Note. Recall that Theorem 5.4, "A Criterion for Diagonalization," states that $n \times n$ matrix A is diagonalizable if and only if the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity. This motivates the following *definition* of diagonalizable for a linear transformation on a finite-dimensional vector space.

Definition 7.2. A linear transformation T of a finite-dimensional vector space V into itself is *diagonalizable* if V has an ordered basis consisting of eigenvectors of T.

Example. Page 407 Number 20.

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