Compass and Straight Edge Constructions

A Supplement to Dr. Bob's Introduction to Modern Algebra 2 Class Notes

SLIDE. The rules by which compass and straight edge constructions are performed are as follows. We start with a line segment of a length which we define as the unit length. For example, we could assume the line segment lies along the x-axis between the points $(0, 0)$ and $(1, 0)$. We can construct other lines or line segments using a straight edge through two constructed points. Given a point p and a line segment of a given length ℓ, we can use the compass to construct a circle with center p and radius ℓ. Given a line segment of a certain length, a line segment of the same length can be constructed on any given line. A point is *constructed* when it results from the intersection of two lines, two circles, or a line and a circle.

SLIDE. Given two points, we place the straight edge **(enter)** so that a line (or line segment) can be drawn **(enter)** through the two points.

SLIDE. Given two points, we can place the compass on the two points **(enter)** and can create a circle **(enter)**.

SLIDE. Euclid's *Elements of Geometry* depends, philosophically at least, very heavily on the idea of compass and straight edge constructions. In fact, the very first result, Proposition 1 of Book I, is a demonstration of the construction of an equilateral triangle using a compass and straight edge. The proposition states: "On a given finite straight line, to construct an equilateral triangle." This is typical wording for a result in the *Elements*, and reflects the idea that a geometric object does not exist unless it can be constructed. The construction proceeds as follows. First, **(enter)** draw a circle with radius equal to the distance between the two end points of the line segment and centered on the right point. Second, **(enter)** draw a circle with the same radius centered on the left point. **(enter)** Find a point of intersection of the two circles, and connect **(enter)** the three resulting points. Since each line segment in the resulting triangle is a radius of one of the circles, then the triangle is equilateral.

SLIDE. In our first theorem, we prove that a given line segment can be bisected. As in the previous result, construct a circle **(enter)** centered on each endpoint of the line segment with radius equal to the distance between the two points. **(enter)** Find the points **(enter)** where the circles intersect. Join these two points **(enter)** with a new line, and the new line then intersects the original line segment at the midpoint of the line segment. As an additional observation, **(enter)** we see that this construction also allows us to construct a perpendicular line to a given line segment through the midpoint of the line segment. More generally, the perpendicular can similarly be constructed through any point of the line segment.

SLIDE. We can use the previous theorem to show that any angle can be bisected. First, **(enter)** use the compass to construct an arc through both rays of the angle. Find the two points **(enter)** of intersection of the arc with the rays of the angle. Now, join **(enter)** these two points to produce a cord for the angle. By Theorem 1, **(enter)** the cord can be bisected. We then draw a ray **(enter)** through the vertex of the original angle and the point of bisection just produced. This results in a bisection of the original angle.

SLIDE. We now address perpendiculars. Namely, given a line and a point not on the line, a perpendicular to the line passing through the point can be constructed. First, **(enter)** draw a circle centered at the given point with a radius sufficiently large as to intersect the given line in two points. The two points of intersection **(enter)** determine a line segment. By Theorem 1, this line segment can be bisected. **(enter)** The line through the original point **(enter)** and the midpoint of the new line segment is perpendicular to the original line.

SLIDE. Given a line and a point not on the line, a parallel to the line passing through the point can be constructed. By Theorem 3, we can construct a perpendicular **(enter)** to the given line through the given point. Now, **(enter)** translate the distance from the original point to the original line to find a new point above the original point on the perpendicular, as shown. Since the black point is the midpoint of the line segment through the two blue points, by Corollary 1 we can construct a perpendicular to the new line which passes through the black point. **(enter)** The new line is parallel to the original line and passes through the given point.

SLIDE. We now give a result which we will generalize later. A line segment can be trisected (that is, broken into three parts of the same length). First, **(enter)** we draw a new line through one end point of the given line segment to produce an angle (we know that, for example, a 60° angle can be constructed by Proposition 1 of Euclid). On this new line, **(enter)** we mark off the given unit three times. **(enter)** We connect the third green point **(enter)** to the other end point of the given line segment and call the resulting line ℓ, as shown. We now draw parallels

(enter) to line ℓ through the first and second green points using Theorem 4. The points of intersection of these three red lines then determine points on the original line segment which result in the trisection of the original line segment.

SLIDE. Our textbook claims in Theorem 32.1 that if α and β are constructible numbers, then so are $\alpha+\beta$, $\alpha-\beta$, $\alpha\beta$, and α/β if $\beta \neq 0$. If this can be demonstrated, then we can show as a corollary that the constructible real numbers form a field. As we will see, the field of constructible numbers are a field "between" the rational numbers and the real numbers. As a parenthetic remark, the constructible numbers are in fact a field between the rationals and the algebraic numbers. We now go through a proof of Theorem 32.1.

SLIDE. If α and β are constructible real numbers, then so is $\alpha+\beta$. We simply take the segment of length β and translate **(enter)** it onto the end of a line segment with distance α marked off on it. **(enter)** This results in a line segment of the desired length.

SLIDE. If α and β are constructible real numbers, then so is $\alpha - \beta$. As in the previous result, we simply take the segment of length β and translate **(enter)** it onto the line segment with distance α marked off on it. **(enter)** This results in a segment of the desired length, as shown. For the sake of illustration, we assume $\beta < \alpha$, but this is not necessary.

SLIDE. If α and β are constructible real numbers, then so is $\alpha\beta$. This result requires the use of properties of similar triangles. Mark off distance α on a line segment as shown. **(enter)** Introduce another line segment through point O as shown. **(enter)** Mark off a unit distance along the new green line from point O and introduce point P. **(enter)** Similarly, mark off distance β along the green line and determine point B. For the sake of illustration, we assume $\beta > 1$. **(enter)** Connect point P and point A. Draw a parallel to line PA through point B, **(enter)** which can be done by Theorem 4. Denote the intersection of this new line with the extended original line segment as Q. **(enter)** Now triangle OPA is similar to triangle OBQ. **(enter)** Therefore, $\dfrac{1}{\alpha} = \dfrac{\beta}{|\overline{OQ}|}$. **(enter)** Hence, $\alpha\beta = |\overline{OQ}|$. **(enter)** This gives the desired result.

SLIDE. If α and β are constructible real numbers, then so is α/β, provided $\beta \neq 0$. This construction is similar to the previous one. **(enter)** As before, introduce a new line and mark off distances one and β. **(enter)** Connect points A and B. **(enter)** Now construct a parallel to the line through A and B which passes through point P, which can be done by Theorem 4. Denote as point Q the point of intersection as shown. We again use a similar triangle argument. **(enter)** Triangle OPQ is similar to triangle OBA. **(enter)** Therefore, $\dfrac{|\overline{OQ}|}{1} = \dfrac{\alpha}{\beta}$. **(enter)** Hence, $\alpha/\beta = |\overline{OQ}|$. **(enter)** This gives the desired result.

SLIDE. Theorem 32.1 shows that all rational numbers are constructible. We now give a result which shows that more numbers are constructible than just rationals. If α is a positive constructible real number, then so is $\sqrt{\alpha}$. Let line segment OA have length α where, for the sake of illustration, $\alpha > 1$. **(enter)** Extend segment OA by length 1 and introduce point P as shown. **(enter)** Find the midpoint of line segment AP. **(enter)** With this midpoint as the center, introduce a circle with diameter AP. **(enter)** Construct a perpendicular to segment AP through point O, which can be done by Theorem 3. Denote the point of intersection of this perpendicular with the circle as Q, as shown. **(enter)** Join points P and Q, and points A and Q to produce a triangle. **(enter)** We know that triangles QAP, OPQ, and OAP are right triangles. **(enter)** Therefore, angles OQP and OQA are complementary. **(enter)** Also angles OPQ and OAQ are complementary. **(enter)** Therefore, triangle OPQ is similar to triangle OQA. **(enter)** We now have $\frac{|\overline{OQ}|}{|\overline{OA}|} = \frac{|\overline{OP}|}{|\overline{OQ}|}$. **(enter)** Equivalently, $|\overline{OQ}|^2 = |\overline{OP}||\overline{OA}|$. **(enter)** Since the length of segment OP is 1, then $|\overline{OQ}|^2 = |\overline{OA}|$. Therefore, since the length of OA is α, we have $|\overline{OQ}|^2 = \alpha$, **(enter)** or that $\sqrt{\alpha} = |\overline{OQ}|$, and the result follows.

SLIDE. We now have that the set of constructible real numbers forms a subfield of the real numbers. We will see in class in Theorem 32.6 that the field of constructible real numbers consists precisely of all real numbers that we can obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations.

SLIDE. We now show that an angle θ is constructible if and only if the length $|\cos\theta|$ is constructible. **(event)** First, suppose the length $|\cos\theta|$ is constructible. **(event)** Construct a perpendicular to the line segment of length $|\cos\theta|$ where the perpendicular intersects the line segment at the right hand endpoint, which can be done by Corollary 1. **(event)** Construct a circle of radius 1 centered on the left end point of the line segment. **(event)** Connect the left end point of the line segment to the point of intersection of the perpendicular and the circle. **(enter)** We now have a right triangle which contains the acute angle θ. If θ is not acute, then we can construct it using the angle constructed here and a combination of right angles.

SLIDE. **(enter)** Now suppose angle θ is constructible and let point O be the vertex of the angle. **(enter)** Construct a circle of radius 1 and center O. **(enter)** From the point of intersection of the circle and one of the rays of angle θ, construct a perpendicular to the other ray of θ, which can be done by Theorem 3. **(enter)** Therefore, the distance $|\cos\theta|$ can be constructed, as shown.

SLIDE. We now explore the construction of regular n-gons. We start by constructing a regular pentagon. Suppose we are given a circle, the center of the circle, and a point on the circle. **(enter)** Construct a diameter of the circle through the two given points. **(enter)** Construct a perpendicular to the diameter by Corollary 1 and find the point where the perpendicular intersects the circle. **(enter)** Bisect the radius of the circle as shown, which can be done by Theorem 1. **(enter)** Connect the midpoint of the radius and the point on the circle, as shown. **(enter)** Bisect the angle as shown using Theorem 2 and find the point of intersection as shown in blue. **(enter)** Construct a perpendicular to the radius through the blue point as shown, using Corollary 1. Find the point of intersection of this perpendicular with the circle. **(enter)** Now connect the two points as shown.

SLIDE. We now repeat this process **(enter)** by reflecting the process to the left side of the vertical radius.

SLIDE. Repeat the process **(enter)** three more times to get a regular pentagon.

SLIDE. The construction of a regular hexagon is easier than the construction of a regular pentagon. Suppose a circle is given with the center and a point on the circle identified as shown. **(enter)** Construct a new circle centered at the point on the original circle with the same radius as the original circle. **(enter)** Find the two points of intersection of the two circles. **(enter)** Construct three diameters of the original circle, as shown. Mark the points of intersection of these diameters with the circle. **(enter)** Connect the points on the original circle to produce a hexagon.

SLIDE. In Section 55 of Fraleigh, we will classify exactly which regular n-gons are constructible. A *Fermat prime* is a prime number of the form $2^{(2^k)} + 1$ where k is a non-negative integer. The only known Fermat primes correspond to $k = 1$, 2, 3, and 4 and are (respectively) 3, 5, 17, 257, and 65,537. Theorem 55.8 of Fraleigh states: "The regular n-gon is constructible with a compass and a straight edge if and only if all the odd primes dividing n are Fermat primes whose squares do not divide n." Notice that this implies that a regular n-gon is not constructible for $n = 7$, 9, 11, 13, 14, 18, and 19 (for example). Constructions for regular n-gons for $n \leq 20$ were given (or implied) in Euclid's *Elements* for all possible cases, except $n = 17$. Carl Gauss is the first to show that a regular 17-gon could be constructed. In fact, it was Gauss who first gave the sufficiency for Theorem 55.8 (which was proven in its entirety in 1837 by Pierre Wantzel).

SLIDE. We now address the three geometric constructions from classical Greece. First, doubling the cube is impossible. **(even)** That is, given a side of a cube, it is not always possible to construct with a compass and straight edge the side of a cube that has double the volume. **(even)** This is true because, by Theorem 32.6, $\sqrt[3]{2}$ is not a constructible number.

SLIDE. Second, trisecting an angle is impossible. That is, there exists an angle that cannot be trisected with a straight edge and compass. **(enter)** This result might seem a bit surprising, because we can bisect an angle, by Theorem 2, and the construction involves bisecting a line segment, which was accomplished in Theorem 1. We can also trisect a line segment by Theorem 5. **(enter)** However, there is no general straight edge and compass construction to trisect an angle. **(enter)** In fact, a 60° angle cannot be trisected (and therefore a 20° angle is not constructible). This is because the cosine of 20° satisfies the equation $4\cos^3 20^\circ - 3\cos 20^\circ = \frac{1}{2}$. The solutions to this cubic equation are not constructible (they involve cube roots which are not constructible by Theorem 32.6) and so $\cos 20^\circ$ is not constructible. Therefore, by the lemma to Theorem 32.11, 20° is not a constructible angle.

SLIDE. Thirdly, squaring a circle is impossible. **(even)** That is, given a circle it is not always possible to construct with a compass and straight edge a square with area equal to the area of the given circle. **(even)** Performing this construction would require constructing a line segment of length $\sqrt{\pi}$. Since π is not constructible (π is not even an algebraic number—it is transcendental), then $\sqrt{\pi}$ is not constructible and hence squaring the circle is not possible.

SLIDE. References. Of particular interest, are Chapters 32 and 55 of John B. Fraleigh's *A First Course in Abstract Algebra*, 7th edition.