Introduction to Algebra, MATH 5127

Homework 10, Sections IV.19, IV.20, IV.21, Solutions

Due Wednesday November 26, 2014 at noon

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. **Justify all steps** by quoting the relevant results from the textbook.

IV.19.12 Let $\langle R, +, \cdot \rangle$ be a commutative ring with unity and characteristic 3. Since R is commutative, the Binomial Theorem holds. Use it to compute and simplify $(a+b)^9$ for $a, b \in R$.

IV.19.A. Let $\langle R, +, \cdot \rangle$ be a commutative ring with nonzero $a, b \in R$ and let ab be a zero divisor. Prove that either a or b is a zero divisor.

IV.19.B. Let $\langle R, +, \cdot \rangle$ be a finite commutative ring with no zero divisors and with at least two elements. Prove that R has unity. HINT: Let $R = \{0, r_1, r_2, \ldots, r_n\}$ where $r_i \neq 0$. Show $r_1 R = R$. Conclude that some r_j is unity.

IV.20.5. Use the Little Theorem of Fermat to find the remainder when 37^{49} is divided by 7. Explain your computations.

IV.20.9. Compute $\varphi(pq)$ where p and q are distinct primes. Is $\varphi(pq) = \varphi(p^2)$ when $q = p$?

IV.20.27. Show that 1 and $p-1$ are the only elements of the field $\mathbb{Z}_p$ (p prime) that are their own multiplicative inverse. HINT: Explain why a being its own inverse is equivalent to a being a solution to $x^2 - 1 = 0$. Find all solutions to $x^2 - 1 = 0$ in $\mathbb{Z}_p$.

IV.21.1. Let $D = \{n + mi \mid n, m \in \mathbb{Z}\}$. This is an integral subdomain of $\mathbb{C}$ called the *Gaussian integers*. With the notation of page 191, describe the sets $D \times D$ and S. What are the equivalence classes of the set S? What are the elements of $i[D]$? HINT: To find the quotient of two complex numbers (the elements of $i[D]$), you will need to use complex conjugates (see equation (7) on page 15).

IV.21.13. Use Exercise 21.12 to prove that any commutative ring containing an element $a \neq 0$, where a is not a divisor of zero, can be enlarged to a commutative ring with unity. HINT: Let $T = \{a^n \mid n \in \mathbb{N}\}$. Show that T satisfies the hypotheses of Exercise 21.12. Show that the resulting ring is commutative.

IV.21.A. Bonus. Let D be an integral domain and let F be the field of quotients of D. Prove that if E is any field that contains D, then E contains a subfield that is ring-isomorphic to F. (Thus, the field of quotients of an integral domain D is the smallest field containing D.)