Introduction to Algebra, MATH 5127

Homework 2, Sections I.2 and I.3

Due Friday September 12, 2014 at 2:30

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting the relevant results from the textbook.

I.2.A. Determine whether the binary operation $a * b = \sin(a+b)$ on $\mathbb{R}$ is commutative and whether it is associative. If the property holds, give a proof. If the property does not hold, give a counterexample.

I.2.23b. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$. Is H closed under matrix multiplication? If so, give a (general) proof. If not, a specific example showing violation of closure is sufficient.

I.2.36. Suppose that $*$ is an associative binary operation on a set S. Let $H = \{a \in S \mid a * x = x * a \text{ for all } x \in S\}$. Prove that H is closed under $*$. (We think of H as consisting of all elements of S that commute with every element of A.)

I.2.37. Suppose that $*$ is an associative and commutative binary operation on a set S. Show that $H = \{a \in S \mid a * a = a\}$ is closed under $*$. (The elements of H are *idempotents* of the binary operation $*$.)

I.3.8. Determine whether $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ where $\phi(A)$ is the determinant of matrix A, is an isomorphism between $\langle M_2(\mathbb{R}), \cdot \rangle$ and $\langle \mathbb{R}, \cdot \rangle$. If so, then prove ϕ is an isomorphism and if not then explain which part of the definition of isomorphism is violated. Recall that an isomorphism is a one to one and onto mapping with the homomorphism property. HINT: For all $A, B \in M_2(\mathbb{R})$ we have $\det(AB) = \det(A)\det(B)$.

I.3.27. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$ and $\psi : S' \to S''$ is an isomorphism of $\langle S', *' \rangle$ with $\langle S'', *'' \rangle$, then the composite function $\psi \circ \phi$ is an isomorphism of $\langle S, * \rangle$ with $\langle S'', *'' \rangle$. You may assume that $\psi \circ \phi$ is one to one and onto, and so you only need to show that the homomorphism property is satisfied.

I.3.30. Prove that the associativity of a binary operation is a structural property of a binary algebraic structure. That is, prove that if $*$ is associative in $\langle S, * \rangle$ and $\phi : \langle S, * \rangle \to \langle S', *' \rangle$ is an isomorphism, then $*'$ is associative in $\langle S', *' \rangle$.