Introduction to Algebra, MATH 5127

Homework 3, Sections I.4 and I.5

Due Friday September 19, 2014 at 2:30

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. **Justify all steps** by quoting the relevant results from the textbook.

I.4.A. The elements $\{1, 2, 3, 4\}$ form a group under multiplication modulo 5, $\times_5$. Make a multiplication table (or Cayley table) of this group. There are two groups with four elements, $\mathbb{Z}_4$ and the Klein-4 group (see page 6 of the class notes for Section I.4). To which of these is $\langle\{1, 2, 3, 4\}, \times_5\rangle$ isomorphic? What is the isomorphism and why (illustrate with another multiplication table)? HINT: Define the isomorphism ϕ by giving its values on $1, 2, 3, 4$. Denote the elements of $\mathbb{Z}_4$ as $0', 1', 2', 3'$. Denote the elements of the Klein-4 group as $1, \alpha, \beta, \gamma$ as in the class notes.

I.4.B Let G be a group such that for all $a, b, c \in G$, if $a * b = c * a$ then $b = c$. Prove that group G is abelian. HINT: Let $g_1, g_2 \in G$ and cleverly choose values of a, b, c to show $g_1 * g_2 = g_2 * g_1$.

I.4.32. Show that every group G with identity e and such that $x * x = e$ for all $x \in G$ is abelian. HINT: Consider $(a * b) * (a * b)$.

I.5.28. List the elements of the cyclic subgroup of group V generated by element c where the group table for V is given in Table 5.11 on page 51. Show your work!

I.5.52. Let S be a subset of a group G. Define $H_S = \{x \in G \mid xs = sx \text{ for all } s \in S\}$. Prove that H_S is a subgroup of G. For group G, the subgroup H_G and is called the *center* of group G. Prove that H_G is abelian.

I.5.A. Bonus. Let G be an abelian group and let $g_1, g_2 \in G$. Prove that the set $\langle g_1, g_2\rangle = \{g_1^i g_2^j \mid i, j \in \mathbb{Z}\}$ is a subgroup of G. Use Theorem I.5.14.

I.5.B. Let G be a group and H a subgroup of G. For $g \in G$, define $gH = \{gh \mid h \in H\}$. Suppose G is abelian and $g^2 = e$. Prove that the set $K = H \cup gH$ is a subgroup of G. Use Theorem I.5.14.