Graph Automorphism Groups

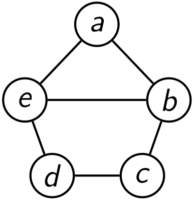
Robert A. Beeler, Ph.D.

East Tennessee State University

February 23, 2018

What is a graph?

A graph $G = (V, E)$ is a set of vertices, V, together with as set of edges, E. For our purposes, each edge will be an unordered pair of distinct vertices.



$V(G) = \{a, b, c, d, e\}$
$E(G) = \{ab, ae, bc, be, cd, de\}$

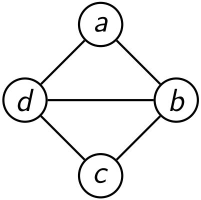
Graph Automorphisms

A graph *automorphism* is simply an isomorphism from a graph to itself. In other words, an automorphism on a graph G is a bijection $\phi : V(G) \to V(G)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G)$.

Note that graph automorphisms *preserve adjacency*. In layman terms, a graph automorphism is a symmetry of the graph.

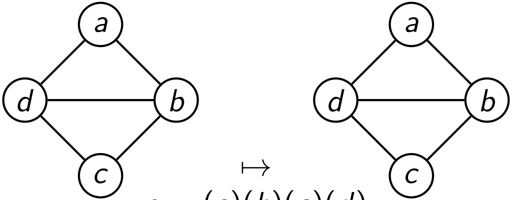
An Example

Consider the following graph:



An Example (Part 2)

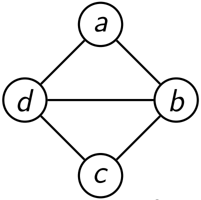
One automorphism simply maps every vertex to itself. This is the *identity* automorphism.



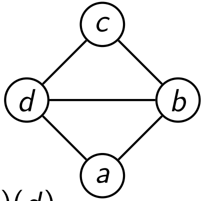
$$e = (a)(b)(c)(d)$$

An Example (Part 3)

One automorphism switches vertices *a* and *c*.



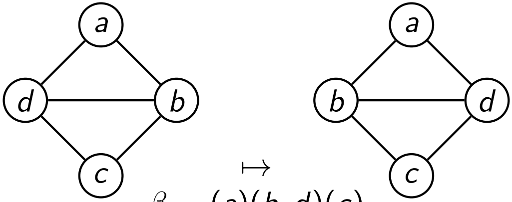
$\mapsto$



$$\alpha = (a, c)(b)(d)$$

An Example (Part 4)

One automorphism switches vertices b and d.

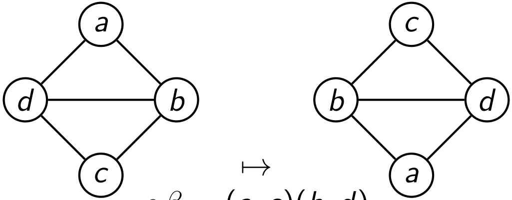


$$\mapsto$$

$$\beta = (a)(b, d)(c)$$

An Example (Part 5)

The final automorphism switches vertices a and c and also switches b and d.



$$\alpha\beta = (a, c)(b, d)$$

The Automorphism Group

Let $Aut(G)$ denote the set of all automorphisms on a graph G. Note that this forms a *group* under function composition. In other words,

(i) $Aut(G)$ is closed under function composition.

(ii) Function composition is associative on $Aut(G)$. This follows from the fact that function composition is associative in general.

(iii) There is an identity element in $Aut(G)$. This is mapping $e(v) = v$ for all $v \in V(G)$.

(iv) For every $\sigma \in Aut(G)$, there is an inverse element $\sigma^{-1} \in Aut(G)$. Since σ is a bijection, it has an inverse. By definition, this is an automorphism.

Facts about graph automorphisms

Graph automorphisms are *degree preserving*. In other words, for all $u \in V(G)$ and for all $\phi \in Aut(G)$, $deg(u) = deg(\phi(u))$.

WHY? Let $u \in V(G)$ with neighbors u_1,...,u_k. Let $\phi \in Aut(G)$. Since ϕ preserves adjacency, it follows that $\phi(u_1)$,...,$\phi(u_k)$ are neighbors of $\phi(u)$. Thus, $deg(\phi(u)) \geq k$. If $v \notin \{u_1, ..., u_k\}$ is a neighbor of $\phi(u)$, then $\phi^{-1}(v)$ is a neighbor of u. Therefore, the neighbors of $\phi(u)$ are precisely $\phi(u_1)$,...,$\phi(u_k)$. Ergo, $deg(u) = deg(\phi(u))$.

Facts about graph automorphisms (Part 2)

Graph automorphisms are *distance preserving*. In other words, for all $u, v \in V(G)$ and for all $\phi \in Aut(G)$, $d(u, v) = d(\phi(u), \phi(v))$.

WHY? Let $u = u_0, u_1, ..., u_{d-1}, u_d = v$ be a shortest path from u to v. Note that, $\phi(u) = \phi(u_0), \phi(u_1), ..., \phi(u_{d-1}), \phi(u_d) = \phi(v)$ is a path from $\phi(u)$ to $\phi(v)$. Thus, $d(\phi(u), \phi(v)) \leq d = d(u, v)$. Suppose that $\phi(u), v_1, ..., v_{m-1}, \phi(v)$ is a shortest path from $\phi(u)$ to $\phi(v)$. It follows that $u, \phi^{-1}(v_1), ..., \phi^{-1}(v_{m-1}), v$ is a shortest path form u to v. It follows that $d(u, v) \leq d(\phi(u), \phi(v))$. Hence, we have equality.

Facts about graph automorphisms (Part 3)

The automorphism group of G is equal to the automorphism group of the complement $\overline{G}$.

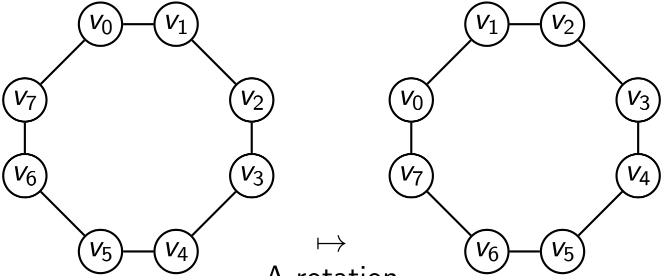
WHY? Note that automorphisms preserve not only adjacency, but non-adjacency as well. Hence, $\phi \in Aut(G)$ if and only if $\phi \in Aut(\overline{G})$. It follows that $Aut(G) = Aut(\overline{G})$.

Graph Families - the Path

By the above comments, v_0 can only map to itself or to the other endpoint v_{n-1}. Thus the only automorphisms are the identity and $\phi(v_i) = v_{n-i-1}$. Hence, $Aut(P_n) \cong \mathbb{Z}_2$.

Graph Families - Cycles

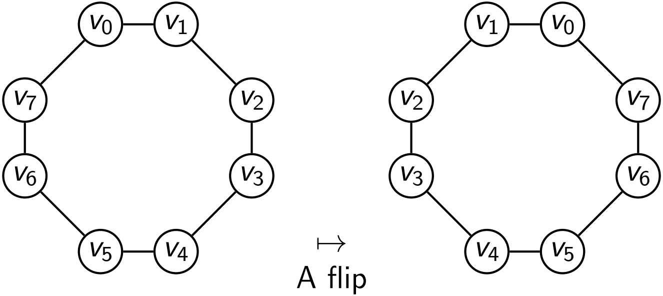
The automorphism group of the cycle C_n is generated by two operations:



A rotation

Graph Families - Cycles

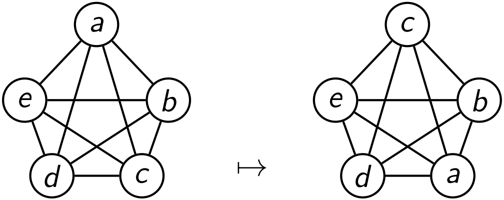
The automorphism group of the cycle C_n is generated by two operations:



So, $Aut(C_n) \cong D_n$, the nth dihedral group.

Graph Families - the Complete Graph

Note that any transposition of vertices is possible on the complete graph. Ergo, $Aut(K_n) \cong S_n$, the nth symmetric group.

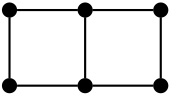


Graph Families - Complete Bipartite

For the complete bipartite graph, $K_{n,m}$, we get all permutations on X and all permutations on Y. If $n = m$, then we also get the permutation that switches x_i with y_i for all i. Thus, if $n \neq m$, then $Aut(K_{n,m}) \cong S_n \times S_m$. If $n = m$, then $Aut(K_{n,m}) \cong S_n^2 \ltimes \mathbb{Z}_2$.

The Cartesian product

Recall that the *Cartesian product* of graphs G and H is the graph with vertex set $\{(g,h) : g \in V(G), h \in V(H)\}$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either *(i)* $g_1 = g_2$ and $h_1 h_2 \in E(H)$ *or (ii)* $h_1 = h_2$ and $g_1 g_2 \in V(G)$. This graph is denoted $G \square H$.



$P_2 \square P_3$

Cartesian Products (Part 2)

$Aut(G) \times Aut(H)$ is a subgroup of $Aut(G \Box H)$.

WHY? Let $\phi \in Aut(G)$ and let $\theta \in Aut(H)$. Consider the mapping $\xi : V(G \Box H) \to V(G \Box H)$ defined by $\xi((g,h)) = (\phi(g), \theta(h))$. We claim that ξ is an automorphism of $G \Box H$. Suppose that (g_1, h_1) and (g_2, h_2) are adjacent in $G \Box H$. If $h_1 = h_2$, then $\theta(h_1) = \theta(h_2)$. Further, g_1 and g_2 would be adjacent in G. It follows that $\phi(g_1)$ and $\phi(g_2)$ are adjacent in G. Since $\xi((g_1, h_1)) = (\phi(g_1), \theta(h_1))$ and $\xi((g_2, h_2)) = (\phi(g_2), \theta(h_2))$, it follows that $\xi((g_1, h_1))$ and $\xi((g_2, h_2))$ are adjacent in $G \Box H$. A similar argument holds if $g_1 = g_2$ and h_1 is adjacent to h_2 in H.

Cartesian Products (Part 3)

A natural question is when $Aut(G\square H)$ contains an element that is not of the form described above. To do this, we need a bit more terminology. A graph D is a *divisor* of a graph G if their exists a graph H such that $G \cong D\square H$. A graph P is *prime* if P has no divisor other than itself and K_1[1]. Graphs G and H are *relatively prime* if they share no common factor other than K_1. With these terms in mind, we present results about the automorphism group of Cartesian products.

[1]Examples of prime graphs include trees, odd cycles, and complete graphs.

Cartesian Products (Part 4)

(i) Every connected graph G can be written as $G \cong G_1 \square \cdots \square G_k$, where the G_i are prime graphs. This factorization is unique, up to permutations on the prime factors[2].
(ii) If G is a connected graph, then $Aut(G)$ is generated by $Aut(G_i)$ and the transpositions interchanging isomorphic prime divisors.
(iii) In particular, if the G_i are relatively prime connected graphs, then $Aut(G)$ is the direct product of the $Aut(G_i)$ over all i.

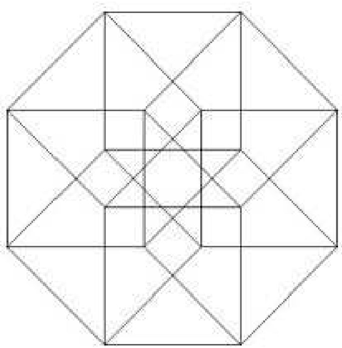
[2] The factorization may not be unique for disconnected graphs. As an example, note that $(K_1 \cup K_2 \cup K_2^2) \square (K_1 \cup K_2^3)$ is isomorphic to $(K_1 \cup K_2^2 \cup K_2^4) \square (K_1 \cup K_2)$.

Cartesian Products (Part 5)

The comment on the previous slide about "transpositions interchanging isomorphic prime divisors" deserves a bit more explanation. Suppose that the connected graph G has prime factorization $G = G_1 \square \cdots \square G_n$, where G_i and G_j are isomorphic prime divisors for some $i \neq j$. Thus, there is isomorphism $\psi : G_i \to G_j$. For all $k \in \{1, ..., |V(G_i)|\}$, suppose that $v_{i,k} \in V(G_i)$ and $v_{j,k} \in V(G_2)$ such that $\psi(v_{i,k}) = v_{j,k}$. Then there is an automorphism $\zeta \in Aut(G_i \square G_j)$ such that $\zeta = (v_{j,1}, v_{i,1})...(v_{j,|V(G_2)|}, v_{i,|V(G_1)|})$. In other words, ζ "swaps" the isomorphic prime factors G_i and G_j.

Cartesian Products (Part 6)

In particular, for the hypercube Q_n, $Aut(Q_n) \cong \mathbb{Z}_2^n \ltimes S_n$.



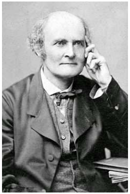
The Inverse Problem

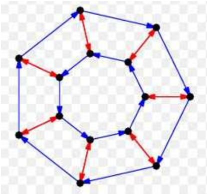
Earlier in this talk, we have concentrated on the following problem: "Given a graph G, determine the automorphism group of G."
In 1936, Dénes König (1884-1944) proposed the following problem his book *Theorie der endlichen und unendlichen Graphen*[3]: "Given a finite group Γ, find a graph G such that $Aut(G) \cong \Gamma$."

This problem was solved by Roberto Frucht (1906-1997) in 1939.

Cayley Graphs

Frucht's construction makes heavy use of Cayley graphs. These were introduced by Arthur Cayley (1821-1895) in 1878. Suppose that Γ is a group and that S is a generating set of Γ. $Cay(\Gamma, S)$ has a vertex for each element of the group Γ. Let $x, y \in V(Cay(\Gamma, S)$. There is a arc pointing from x to y if and only if there exists an $g \in S$ such that $xg = y$. Traditionally, the different generators are represented by different colored arcs.





Frucht's Theorem

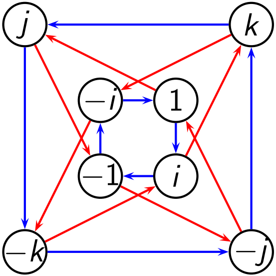
The idea behind the proof of Frucht's Theorem is quite simple

(i) Begin with the Cayley graph $Cay(\Gamma, S)$.

(ii) Replace each arc of color c_i with a graph g_i that preserves the orientation, but does not introduce any new symmetry.

Example - The Quaternion Group

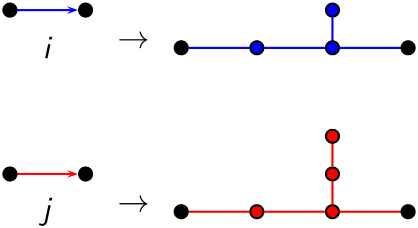
As an example, take the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = ijk = -1\}$. A generating set for this group is $S = \{i, j\}$.



The Cayley Graph $Cay(Q_8, \{i, j\})$

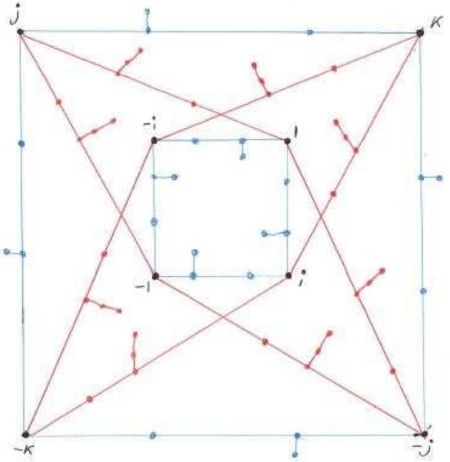
Example - The Quaternion Group (Part 2)

We then replace each of the arcs as follows:



Example - The Quaternion Group (Part 3)

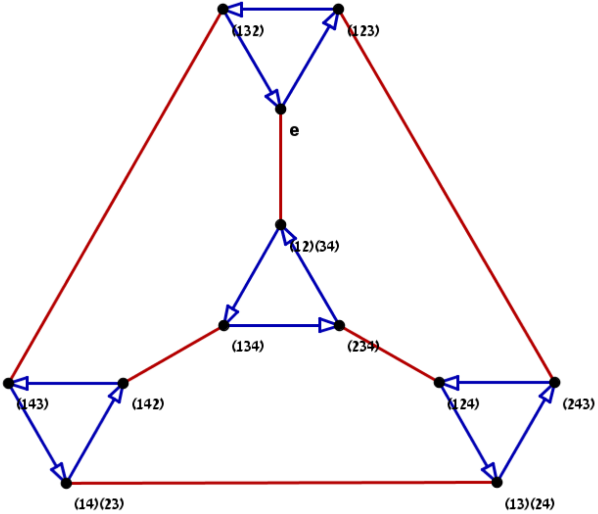
The result is the following:



Example - The Alternating Group

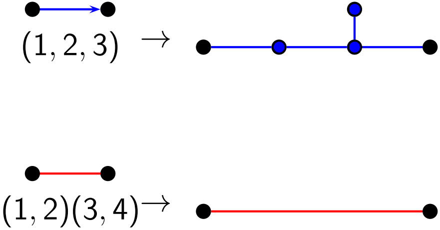
Consider the Consider the Alternating group A_4. This is the group of order 12 consisting of all even permutations on the set $\{1, 2, 3, 4\}$. This group is generated by $(1, 2, 3)$ and $(1, 2)(3, 4)$. We represent right multiplication by $(1, 2, 3)$ as a blue arc. We represent right multiplication by $(1, 2)(3, 4)$ as a red edge. Note that since $(1, 2)(3, 4)$ is its own inverse, the red edges are undirected.

Example - The Alternating Group (Part 2)



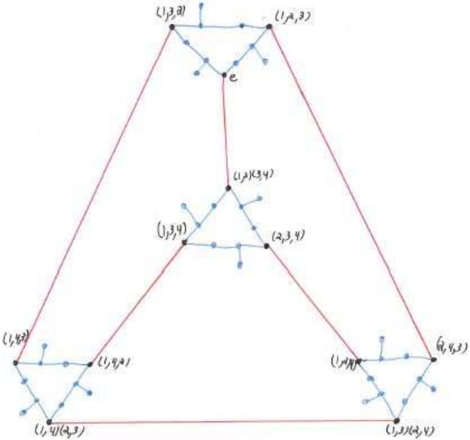
Example - The Alternating Group (Part 3)

We then do our replacements as we did above.



Example - The Alternating Group (Part 4)

Our result:



Other Frucht-type constructions

As it turns out, graphs are rather pliable things. Frucht's theorem still holds, even if we put additional restrictions on our graphs. For example:

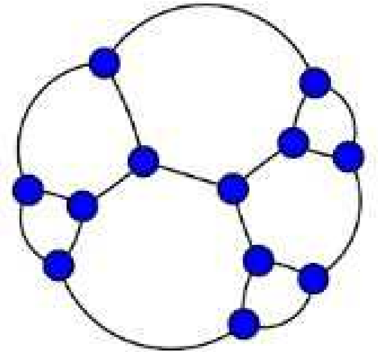
(i) Given a finite group Γ, there is a k-regular graph G such that $Aut(G) \cong \Gamma$.

(ii) Given a finite group Γ, there is a k-vertex-connected graph G such that $Aut(G) \cong \Gamma$.

(iii) Given a finite group Γ, there is a k-chromatic graph G such that $Aut(G) \cong \Gamma$.

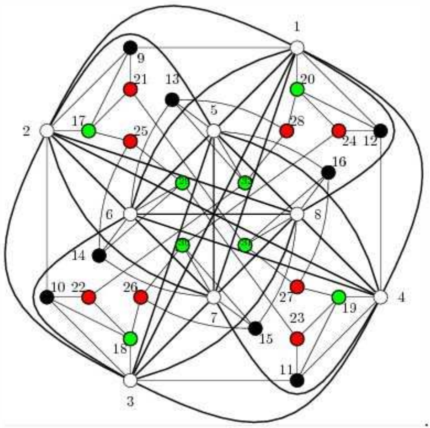
Minimum graphs with a given group

A natural problem is given a group Γ, find the minimum graph whose automorphism group is isomorphic to Γ. For example, below we have the Frucht graph. This is the smallest 3-regular graph whose automorphism group is the trivial group.



Minimum graphs with a given group (Part 2)

Just for fun, below is the smallest graph whose automorphism group is the quaternion group. This graph has 32 vertices.



Other Problems - Edge Automorphisms

Throughout this talk, we have looked at *vertex automorphisms*. We can just as easily talk about *edge automorphisms*.

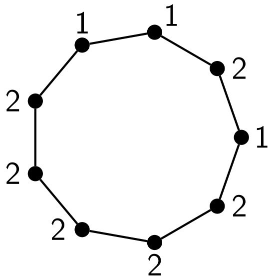
An *edge automorphism* is a bijection $\xi : E(G) \to E(G)$ such that the edges x and y share and endpoint in G if and only if the edges $\xi(x)$ and $\xi(y)$ share an endpoint in G. Again, the set of edge automorphisms forms a group under function composition. A natural question is to determine the edge automorphism group of a given graph.

Other Problems - Distinguishing Labelings

In 1996, Albertson and Collins introduced the *distinguishing number of a graph*. For the distinguishing number, we label the vertices of G with (not-necessarily distinct) elements of $\{1, ..., k\}$. The goal is to do this in such a way that no element of $Aut(G)$ preserves all of the vertex labels. However, we wish to do this in such a way that we use the minimum number of labels as possible. This minimum number is the *distinguishing number*.

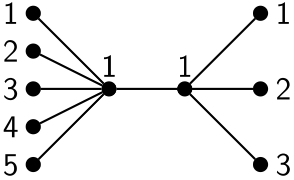
Other Problems - Distinguishing Labelings (Part 2)

As an example, consider the cycle:



Other Problems - Distinguishing Labelings (Part 3)

As a second example, consider the double star:



Other Problems - Distinguishing Chromatic Numbers

In 2006, this was followed by a paper by Collins and Trenk that introduced the *distinguishing chromatic number of a graph*. This is a distinguishing labeling that is also a proper coloring.

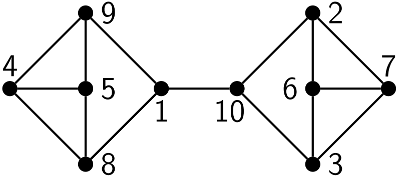
Other Problems - Palindromic Labelings

In December 2017, I submitted a manuscript introducing *palindromic labelings*. A palindromic labeling is a bijection $f : V(G) \to \{1, ..., |V(G)|\}$ such that if $uv \in E(G)$, then there exists $x, y \in V(G)$ such that $xy \in E(G)$, $f(x) = |V(G)| + 1 - f(u)$, and $f(y) = |V(G)| + 1 - f(v)$. A graph that admits a palindromic labeling is a *palindromic graph*. Examples of palindromic graphs include paths, cycles, and complete graphs.

Equivalently, a graph G is palindromic if there exists $\phi \in Aut(G)$ such that ϕ^2 is the identity and ϕ has at most one fixed point.

Other Problems - Palindromic Labelings (Part 2)

An example of a palindromic labeling:



Questions?