Section VI.32. Geometric Constructions

Note. In this section we explore the three famous compass and straight edge constructions from classical Greece:

1. *Doubling the Cube*: For a cube of a given size (i.e., given the length of a side), construct a cube of twice the volume of the given cube.

2. *Squaring the Circle*: For a given circle (i.e., given the diameter of the circle), construct a square with the same area as the circle.

3. *Trisect an Angle*: Given an angle, find an angle 1/3 the size of the given angle.

Surprisingly, none of these are possible and this can be shown using our knowledge of field theory.

Note. An explanation of what it means to perform a compass and straight edge construction, as well as illustrations of the results of this section, are given in my YouTube video Compass Straightedge Constructions (it has over 10,000 views; accessed 3/21/2024).

Definition. A real number α is *constructible* if we can construct a line segment of length $|\alpha|$ in a finite number of steps from a given unit length segment and a straight edge and compass (as described in the supplement).

Theorem 32.1. If α and β are constructible real numbers, then so are $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, and α/β if $\beta \neq 0$.

Note. An animated proof of Theorem 32.1 is given in the YouTube video Compass Straightedge Constructions (accessed 3/21/2024).

Corollary 32.5. The set of constructible real numbers C forms a subfield of the field of real numbers.

Note. Theorem 32.1 implies that each integer is constructible, and so each rational number is constructible: $\mathbb{Q} \leq C$. We now seek to specifically classify the elements of C.

Note. As illustrated in the supplement, compass and straight edge constructions take place in the Euclidean plane. Since $\mathbb{Q} \leq C$, each point in the Cartesian plane with two rational coordinates can be located. The *only* way we can locate other points in the plane is through one of the following intersections of lines and circles:

Case 1. As an intersection of two lines, each of which passes through two known points having rational coordinates;

Case 2. As an intersection of a line that passes through two points having rational coordinates and a circle whose center has rational coordinates and whose radius is rational;

Case 3. As an intersection to two circles whose centers have rational coordinates and whose radii are rational.

Note. We know from classical algebra that the equations of the lines and circles mentioned above are of the form

$$ax + by + c = 0 \text{ and } x^2 + y^2 + dx + ey + f = 0$$

where $a, b, c, d, e, f \in \mathbb{Q}$. We now explore the implications of the three cases mentioned above.

Note. In Case 1, we need to solve two linear equations in two unknowns (x and y). We will get as solutions rational combinations of the coefficients of the two linear equations. In Case 3, if the two circles have equations

$$x^2 + y^2 + d_1x + e_1y + f_1 = 0 \text{ and } x^2 + y^2 + d_2x + e_2y + f_2 = 0,$$

then subtracting these equations leads to the linear equation $(d_1 - d_2)x + (e_1 - e_2)y + (f_1 - f_2) = 0$, which is the equation for the chord passing through the two points of intersection of the circles (or a single point, or no points, depending on the geometry of the situation). So we need to solve the system consisting of this linear equation and one of the equations of a circle. Therefore, case 3 reduces to Case 2.

Note. In Case 2, we have the system

$$ax + by + c = 0 \qquad (1)$$

$$x^2 + y^2 + dx + ey + f = 0. \qquad (2)$$

We can solve for y in (1) and get y as a rational expression of x (namely $y = (-(ax+c)/b)$, and then substitute the result into (2) to produce a quadratic equation in

x which yields values of x based on rational combinations of coefficients and the square root function, as given by the quadratic formula. Therefore, by Theorem 32.1, every rational number can be constructed and by these observations any square roots of rational numbers can be constructed. The process can then be iterated to produce square roots of rational combinations of square roots of rational combinations... a finite number of times.

Note. Fraleigh is a little informal on the passage from $\mathbb{Q}$ to constructible numbers (see the first full paragraph on page 296). For a clearer proof, see pages 238–240 of Thomas Hungerford's *Algebra* (Springer-Verlag, 1974). The constructible numbers can then be described as:

(i) All rational numbers are constructible (as given by Theorem 32.1).

(ii) If $c \geq 0$ is constructible, the $\sqrt{c}$ is constructible (as shown below in Theorem 32.8).

(iii) If c, d are constructible then $c+d$, $c-d$, cd, and c/d for $d \neq 0$ are constructible (as given by Theorem 32.1).

Theorem 32.6. The field of constructible real numbers consists *precisely* of all real numbers that we can obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations.

Proof. The fact that the described numbers are the only ones we can construct follows from the description of the cases above. In the supplement, we show that square roots of positive constructible numbers can be constructed. The theorem then follows. ∎

Corollary 32.8. If γ is constructible and $\gamma \notin \mathbb{Q}$ then there is a finite sequence of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $a_n = \gamma$, such that $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_i)$ is an extension of $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$ of degree 2. In particular, $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2^r$ for some integer $r \geq 0$.

Idea of Proof. As explained above, any constructible number is based on a finite number of extractions of square roots of rational combinations of previously constructible numbers, starting with the rationals $\mathbb{Q}$. The finite sequence of α_i's then follows. Since α_{i+1} is produced from $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_i)$ by taking square roots of a rational combination of elements of $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_i)$, then the extension $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{i+1})$ of $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_i)$ is of degree 2. So $[\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) : \mathbb{Q}] = 2^n$ and by Theorem 31.4

$$2^n = [\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) : \mathbb{Q}] = [\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) : \mathbb{Q}(\gamma)][\mathbb{Q}(\gamma) : \mathbb{Q}].$$

Hence, $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2^r$ for some integer $r \geq 0$. □

Note. We are now equiped, thanks principally to Corollary 32.8, to show that the three classic compass and straight edge constructions cannot be done. This is because each requires the construction of a nonconstructible number.

Theorem 32.9. Doubling the cube is impossible. That is, given a side of a cube, it is not always possible to construct with a compass and straight edge the side of a cube that has double the volume.

Theorem 32.10. Squaring the circle is impossible. That is, given a circle it is not always possible to construct with a compass and straight edge a square with area equal to the area of the given circle.

Theorem 32.11. Trisecting the angle is impossible. That is, there exists an angle that cannot be trisected with a straight edge and compass.

Note. The idea of constructibility was a large part of the geometric ideas of classic Greek geometry. This can be seen by the method of proof in Euclid's *Elements*.

Note. Euclid addresses the construction of regular n-gons specifically as follows:

n	Reference in the *Elements*
3	Book I, Proposition 1
4	Book I, Proposition 46
5	Book IV. Proposition 11
6	Book IV. Proposition 15
15	Book IV. Proposition 16

If a regular n-gon can be constructed and inscribed in a circle (as Euclid does; though he also circumscribes circles as well), then a regular $2n$-gon can be constructed simply by bisecting the edges of the n-gon and projecting the point of bisection onto the circle. With attention on $n \leq 20$, we know that the Greek's could construct n-gons for $n \in \{3, 4, 5, 6, 8, 10, 12, 15, 16, 20\}$. We now know that regular n-gons cannot be constructed for $n \in \{7, 9, 11, 13, 14, 18, 19\}$. Notice that $n = 17$ is in neither list.

Note. Gauss *showed* that a regular 17-gon can be constructed with a compass and straight edge. He did not actually give the construction, but only showed that it existed. He did remark that the key point to the construction is constructing a line of length

$$\frac{1}{16}\left[-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+\sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}-2(1-\sqrt{17})\sqrt{34-2\sqrt{17}}}\right]$$

which we clearly see as a constructible number. The first explicit construction of a 17-gon was given by Ulrich von Huguenin in 1803. H. W. Richmond found a simpler version in 1893. (see page 136 of *Why Beauty is Truth: A History of Symmetry* by Ian Stewart, NY: Basic Books, 2007). It seems surprising that a question addressed in Euclid's *Elements* was picked up in the 19th century and taken further down the field! This accomplishment, again, involves Gauss.

Note. What Gauss did was give sufficient conditions for the construction of a regular n-gon with a compass and straight edge. However, he did not show the conditions were necessary. The problem was completely solved by Pierre Wantzel in 1837 and published as "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas" in *Journal de Mathématiques Pures et Appliquées* **1**(2), 366-372. In this paper he also proved the impossibility of doubling the cube and trisecting and angle (see the historical note on page 298 of Fraleigh).

Note. An n-gon can be constructed with a compass and straight edge if and only if $n = 2^k p_1 p_2 \cdots p_t$ where each p_i is a distinct Fermat prime. A Fermat prime is a prime number of the form $2^{(2^n)} + 1$. The only known Fermat primes are 3, 5, 17, 257, and 65,537. (This information is from Wikipedia.) For more details on this problem, see "The Problem of Constructing Regular Polygons" by B. Bold, Chapter 7 in *Famous Problems of Geometry and How to Solve Them*, NY: Dover, 49–71, 1982.

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