Section VII.38. Free Abelian Groups

Note. In this section, we define "free abelian group," which is roughly an abelian group with a basis. We give examples of such groups and describe properties of the bases. Finally, we give a proof of the Fundamental Theorem of Finitely Generated Abelian Groups (Theorem 11.12).

Note. We shall use additive notation in this section, so for $n \in \mathbb{Z}$ and $x \in G$, nx denotes 0 if $n = 0$, $x + x + \cdots + x$ n-times if $n > 0$, and $(-x) + (-x) + \cdots + (-x)$ $|n|$-times if $n < 0$.

Note. Notice that $\{(1, 0), (0, 1)\}$ is a generating set for group $\mathbb{Z} \times \mathbb{Z}$. Also, each element of $\mathbb{Z} \times \mathbb{Z}$ can be uniquely written in the form $n(1, 0) + m(0, 1)$.

Theorem 38.1. Let X be a subset of a nonzero abelian group G. The following conditions on X are equivalent.

1. Each nonzero element a in G can be expressed uniquely (up to order of summands) in the form $a = n_1x_1 + n_2x_2 + \cdots + n_rz_r$ for $n_i \neq 0$ in $\mathbb{Z}$ and distinct $x_i \in X$.

2. X generates G, and $n_1x_1 + n_2x_2 + \cdots n_rz_r = 0$ for $n_i \in \mathbb{Z}$ and distinct $x_i \in X$ if and only if $n_1 = n_2 = \cdots n_r = 0$.

Definition 38.2. An abelian group having a generating set X satisfying the conditions described in Theorem 38.1 (Condition 1 or Condition 2) is a *free abelian group*. X is a *basis* for the group.

Example 38.3. Notice that $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ (r times) is a free abelian group with basis $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$.

Example 38.4. The group $\mathbb{Z}_n$ is not a free abelian group since $nx = 0$ for every $x \in \mathbb{Z}_n$ and $n \neq 0$ contradicting Condition 2. Also, $\langle \mathbb{Q}, + \rangle$ is not a free abelian group (see Exercise 38.13).

Note. The previous two examples are suggestive of the Fundamental Theorem of Finitely Generated Abelian Groups (Theorem 11.12). Example 38.3 is very suggestive for the structure of a free abelian group with a basis of r elements, as spelled out in the next theorem. The proof is given in Exercise 38.9.

Theorem 38.5. If G is a nonzero (i.e., $G \neq \{0\}$) free abelian group with a basis of r elements, then G is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ for r factors.

Note. The previous theorem and the following theorem are reminiscent of the behavior of vector spaces and their bases.

Theorem 38.6. Let $G \neq \{0\}$ be a free abelian group with a finite basis. Then every basis of G is finite and all bases of G have the same number of elements.

Definition 38.7. If G is a free abelian group, the *rank* of G is the number of elements in a basis for G.

Note. Since a free abelian group with a finite basis has the property that all bases are the same *size*, then Definition 38.7 makes sense for such groups. In fact, for a free abelian group with an infinite basis, all bases are of the same *cardinality*. This is shown in Hungerford's *Algebra* (Theorem II.1.2, page 72).

Note. It is tempting to think of a basis of a vector space as equivalent to a basis of a free abelian group, and to think of the dimension of a vector space as equivalent to the rank of a free abelian group. However, in a vector space there are two operations (vector addition and scalar multiplication), but in an additive group there is only repeated addition. In an n-dimensional vector space, every set of n linearly independent vectors form a basis; but in a free abelian group of rank r, a set of r linearly independent group elements may not form a basis (see Exercise II.1.2(b) of Hungerford's *Algebra* on page 74).

Note. We now turn our attention to the proof of the Fundamental Theorem of Finitely Generated Abelian Groups (Theorem 11.12). We need three preliminary theorems first.

Theorem 38.8. Let G be a finitely generated abelian group with generating set $\{a_1, a_2, \ldots, a_n\}$. Let $\phi : \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \to G$ (where there are n factors of $\mathbb{Z}$) be defined by $\phi(h_1, h_2, \ldots, h_n) = h_1 a_1 + h_2 a_2 + \cdots + h_n a_n$. Then ϕ is a homomorphism onto G.

Theorem 38.9. If $X = \{x_1, x_2, \ldots, x_r\}$ is a basis for a free abelian group G and $t \in \mathbb{Z}$, then for $i \neq j$, the set

$$Y\{x_1, x_2, \ldots, x_{j-1}, x_j + t x_i, x_{j+1}, \ldots, x_r\}$$

is also a basis for G.

Theorem 38.11. Let G be a nonzero free abelian group of finite rank n, and let K be a nonzero subgroup of G. Then K is free abelian of rank $s \leq n$. Furthermore, there exists a basis $\{x_1, x_2, \ldots, x_n\}$ for G and positive integers $d_1, d_2, \ldots, d_s$ where d_i divides d_{i+1} for $i = 1, 2, \ldots, s-1$, such that $\{d_1 x_1, d_2 x_2, \ldots, d_s x_s\}$ is a basis for K.

Note. We now have the equipment to prove the Fundamental Theorem of Finitely Generated Abelian Groups (Theorem 11.12). We give a proof of an initial result first, from which the general theorem follows.

Theorem 38.12. Every finitely generated abelian group is isomorphic to a group of the form

$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

where m_i divides m_{i+1} for $i = 1, 2, \ldots, r - 1$.

Note. Theorem 38.12 gives us the bulk of the Fundamental Theorem of Finitely Generated Abelian Groups. We now state the full theorem and discuss the proof.

Theorem 11.12. Fundamental Theorem of Finitely Generated Abelian Groups.

Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \cdots \times \mathbb{Z}$$

where the p_i are primes, not necessarily distinct, and the r_i are positive integers. The direct product is unique except for possible rearrangement of the factors; that is, the number of factors of $\mathbb{Z}$ is unique (called the *Betti number* of G) and the prime powers $(p_i)^{r_i}$ are unique.

Partial Proof. Theorem 38.12 gives us the form of G in terms of a direct product. By Theorem 11.5 the cyclic groups of Theorem 38.12 can be broken into prime power factors.

Recall that the *torsion subgroup* of abelian group G is the subgroup of G consisting of all elements of G of finite order (see Exercise 11.39, page 112).

From Theorem 38.12, we see that the torsion subgroup of a finitely generated abelian group is the direct product of the various $\mathbb{Z}_n$'s. Let T represent this direct product (appended with copies of $\{0\}$ as needed) and consider G/T (of course, since G is abelian, then T is a normal subgroup of G). Then G/T is of the form $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ for some number of copies of $\mathbb{Z}$. The rank of G/T is the number of copies of $\mathbb{Z}$ in this direct product (for example, one basis for G/T is $\{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)\}$ and all bases are of the same size by Theorem 38.6). So the Betti number (the number of copies of $\mathbb{Z}$) is unique across all such direct product representations (given by Theorem 38.12) of G; the Betti number is the rank of G/T.

The m_i of Theorem 38.12 are called the *torsion coefficients* of G (see Exercise 11.44, page 113). The torsion coefficients of G are shown to be unique in Exercise 38.20 to 38.22.

The uniqueness of the powers of the primes (the prime powers are "peeled off" of the $m)i$, once the uniqueness of the m_i is established above) is given in Exercises 38.14 to 38.19. $\square$

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