2.3. Bolzano-Weierstrass Theorem.

**Note.** In this section we show that every bounded set of real numbers has a “limit point.” We also give a proof of Theorem 2-9 which claims that a sequence of real numbers is Cauchy if and only if it converges.

**Definition.** A real number $x$ is a *limit point* of a set of real numbers $A$ is for all $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon)$ contains infinitely many points of $A$.

**Theorem 2-12.** Bolzano-Weierstrass Theorem.
Every bounded infinite set of real numbers has at least one limit point.

**Theorem 2-13.** Let $\{a_n\}$ be a sequence. Then $L$ is a (finite) subsequential limit of $\{a_n\}$ if and only if $L$ satisfies either of the following:

1. There are infinitely many terms of $\{a_n\}$ equal to $L$, or
2. $L$ is a limit point of a set consisting of the terms of $\{a_n\}$.

**Theorem 2-14.** Every bounded sequence has a convergent subsequence.
2.3. Bolzano-Weierstrass Theorem

Theorem 2-15.

(a) A sequence that is unbounded above has a subsequence that diverges to $+\infty$.

(b) A sequence that is unbounded below has a subsequence that diverges to $-\infty$.

Note. The following theorem gives another characterization of convergent sequences.

Theorem 2-16. A sequence $\{a_n\}$ converges if and only if it is unbounded and has exactly one subsequential limit.

Definition. Let $\{a_n\}$ be a sequence of real numbers. The $\limsup a_n = \overline{\lim} a_n$ is the least upper bound of the set of subsequential limits of $\{a_n\}$, and $\liminf \underline{\lim} a_n$ is the greatest lower bound of the set of subsequential limits of $\{a_n\}$.

Note. By using least upper bounds and greatest lower bounds in the previous definition, we are guaranteed that $\limsup a_n = \overline{\lim} a_n$ and $\liminf \underline{\lim} a_n$ exist for all sequences of real numbers $\{a_n\}$. Of course, these might be $-\infty$ or $+\infty$, however.

Example. If $\{a_n\} = \{\sin n\}$ (n in radians) then $\overline{\lim} a_n = 1$ and $\underline{\lim} a_n = 1$. In fact, the set of subsequential limit points is $[-1, 1]$. 
Exercise 2.3.16. Let \( \{a_n\} \) be a sequence. Then \( \limsup a_n = \lim a_n \) is a subsequential limit of \( \{a_n\} \).

Note. The previous result shows that \( \lim a_n \) is the largest subsequential limit of \( \{a_n\} \). Of course, a similar result holds for \( \lim a_n \).

Theorem 2-17. Let \( \{a_n\} \) be a bounded sequence. Then

(a) \( \lim a_n = L \) if and only if for all \( \varepsilon > 0 \), there exists infinitely many terms of \( \{a_n\} \) in \( (L - \varepsilon, L + \varepsilon) \) but only finitely many terms of \( \{a_n\} \) with \( a_n > L + \varepsilon \).

(b) \( \lim a_n = K \) if and only if for all \( \varepsilon > 0 \), there exists infinitely many terms of \( \{a_n\} \) in \( (K - \varepsilon, K + \varepsilon) \) but only finitely many terms of \( \{a_n\} \) with \( a_n < K - \varepsilon \).

Corollary 2-17. A bounded sequence \( \{a_n\} \) converges if and only if \( \lim a_n = \lim a_n \).

Theorem 2-18.

(a) \( \lim(a_n + b_n) \leq \lim a_n + \lim b_n \), and

(b) \( \lim a_n + \lim b_n \leq \lim(a_n + b_n) \).

Note. Equality does not always hold in Theorem 2-18. Consider \( \{a_n\} = \{\sin^2 n\} \) and \( \{b_n\} = \{\cos^2 n\} \). Then \( \lim a_n = \lim b_n = 1 \), but \( \lim(a_n + b_n) = 1 < a + 1 = 2 \).
Definition. A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be \textit{bounded} if the range of \( f \) is a bounded set. For a bounded function denote \( \text{lub}(\text{range}(f)) \) as \( \sup(f) \) and \( \text{glb}(\text{range}(f)) \) and \( \inf(f) \).

**Theorem 2-19.** Let \( f \) and \( g \) be bounded functions with the same domain. Then:

(a) \( \sup(f + g) \leq \sup(f) + \sup(g) \), and

(b) \( \inf(f) + \inf(g) \leq \inf(f + g) \).

Recall. A sequence \( \{a_n\} \) is a \textit{Cauchy sequence} if

for all \( \varepsilon > 0 \), there exists \( N(\varepsilon) \) such that

if \( n, m > N(\varepsilon) \) then \( |a_n - a_m| < \varepsilon \).

Theorem 2-9 claims that a sequence converges if and only if it is Cauchy. The following exercises verify this claim.

**Exercise 2.3.13.** Let \( \{a_n\} \) be a Cauchy sequence.

(a) Then \( \{a_n\} \) is bounded.

**Proof.** For \( \varepsilon = 1 \), there exists \( N > 0 \) such that for all \( m, n > N \), \( |a_n - a_m| < 1 \), in particular, \( |a_m - a_N| < 1 \) for all \( m > N \). So \( |a_m| < |a_N| + 1 \) for all \( m > N \). Therefore, \( M = \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_N| + 1\} \) is an upper bound for \( \{a_n\} \) and \(-M \) is a lower bound.

(b) There is at least one subsequential limit point for \( \{a_n\} \).
2.3. Bolzano-Weierstrass Theorem

Proof. Since \( \{a_n\} \) is bounded by (a), Theorem 2-14 implies that \( \{a_n\} \) has a convergent subsequence.

(c) There is no more than one subsequential limit point of \( \{a_n\} \).

Proof. Suppose not, suppose \( L \) and \( M \) are both subsequential limit points. WLOG, \( L < M \). Let \( \epsilon = (M - L)/3 \). Then there exists \( N_1(\epsilon) \) such that for all \( k > N_1 \), \( a_k \in (L - \epsilon, L + \epsilon) \). Similarly, there exists \( N_2 \) such that for all \( m > N_2 \), \( a_m \in (M - \epsilon, M + \epsilon) \). But then for \( k \) and \( m \), \( a_k < L + \frac{\epsilon}{3} < M - \frac{\epsilon}{3} < a_m \) from which \( |a_m - a_k| > \epsilon \). So for all \( N > 0 \), there are \( k > N \) and \( m > N \) such that \( |a_m - a_k| > \epsilon \). Therefore, \( \{a_n\} \) is not Cauchy, a contradiction.

(d) \( \{a_n\} \) converges.

Proof. Since there is only one subsequential limit point, \( \lim a_n = \lim a_n \) and by Corollary 2-17, the sequence converges (also follows from Theorem 2-16).

Exercise 2.3.14. A convergent sequence is Cauchy.

Proof. Let \( \epsilon > 0 \). If \( \{a_n\} \) is convergent, then there exists \( N > 0 \) such that for all \( n > N \), \( |a_n - L| < \epsilon/2 \). Let \( n, m > N \). Then \( |a_n - a_m| = |a_n - L - (a_m - L)| \leq |a_n - L| + |a_m - L| = \epsilon/2 + \epsilon/2 = \epsilon \). Therefore \( \{a_n\} \) is Cauchy.

Revised: 1/22/2014