THE FOURTH DIMENSION (AND MORE!)

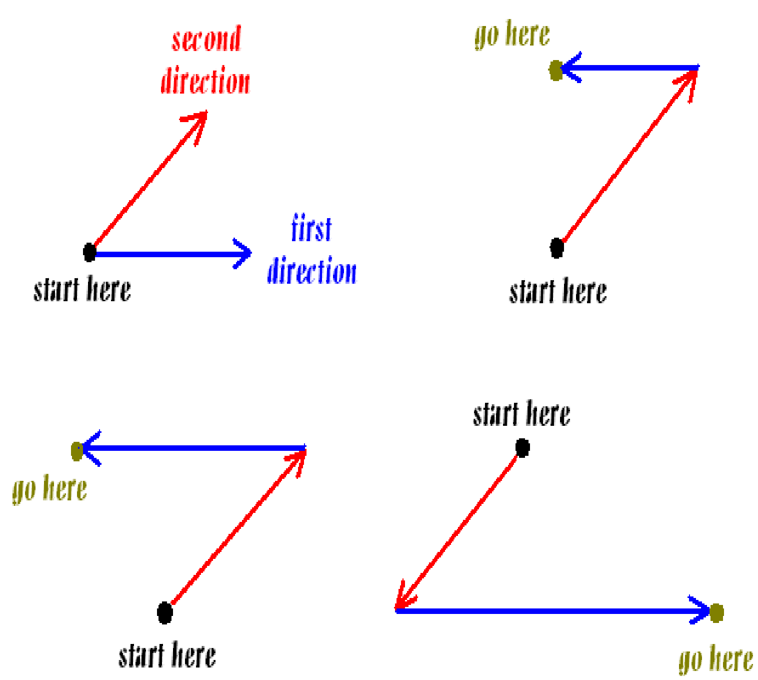
The text leads us from zero dimensions (a point), to one dimension (a line), to two dimensions (a plane), to three dimensions (our familiar space), and then to four dimensions by "inking up" one level and then dragging it to create the next higher level. We approach the same problem by discussing "fundamental directions" and "coordinates."

Fundamental Directions and Dimension

We introduce the idea of fundamental directions and use it to informally define dimension. As the book does, we develop our ideas by building up from a point.

Imagine that we have a car that can move forward and in reverse. However, the steering mechanism has the odd property that it can only point the car in a few (discrete) directions. First, suppose that the steering mechanism is so ill-behaved that you cannot move the car at all. Then you are stuck at a single *point* and cannot travel anywhere. There are *zero directions* in which you can travel and the point forms a zero dimensional space (the word "space" is used since these ideas are explored in detail in the area of linear algebra when addressing the topic of vector spaces). Next, suppose that the steering mechanism will let you point the car in one direction. Then from wherever you start, you can reach any point on a line in the direction in which you can point the car (remember, you can drive both forward and backward). So there is *one direction* in which you can move and so you are stuck in a *one dimensional* space. This is why lines are said to be one dimensional.

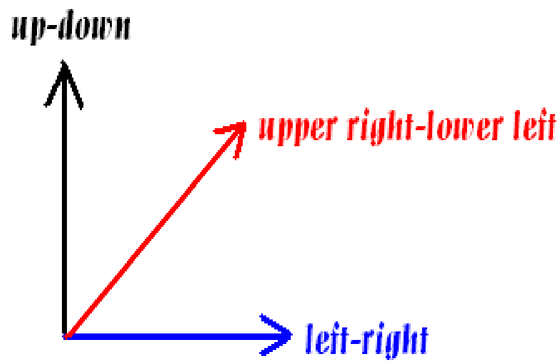
Next, suppose that you can point the car in exactly two different directions (but not opposite directions). In this case, you can reach any point in a plane by driving first in one direction, and then in the second direction:



Question. In the above pictures, suppose the blue direction is directly to the right and the red direction is to the upper right (at a 45° angle to the blue direction). How could we move directly up 10 units? How could we move 5 units to the left and 10 units up?

Questions like this are easier to answer if the two directions are perpendicular (say left-right and up-down). Such a pair of fundamental directions is said to be *orthogonal*.

Next, suppose our imaginary car can travel in three different directions. If all three directions lie in the same plane (i.e., they are coplanar), then we have the same situation as above and the car can travel to any point in the plane. So one of these three directions is redundant and we could make the same trips using only two directions. For example, suppose the car can move left-right, up-down, and upper right-lower left:

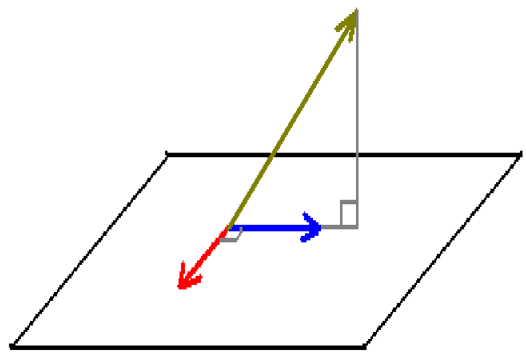


Then any point in the plane can be reached by a combination of the orthogonal directions left-right and up-down. In particular, the upper right-lower left direction is a combination of equal parts of left-right and up-down. If you wanted to travel 10 units to the upper right, say, then you could do so by moving $5\sqrt{2}$ units to the right and $5\sqrt{2}$ units up. The upper right-lower left direction is said to be *linearly dependent* on the left-right and up-down directions. However, we have nothing special against upper right-lower left! We could also omit the up-down direction and could still get anywhere in the plane, as described above. So to travel anywhere in the plane, we must have the ability to move in two different "fundamental directions." Adding more directions in the plane just adds redundancy (linear dependence). So, again, a plane is two dimensional.

Question. Using only left-right and upper right-lower left motion, how would you move 10 units up?

Question. Using only left-right and upper right-lower left motion, how would you move x units to the right and y units up?

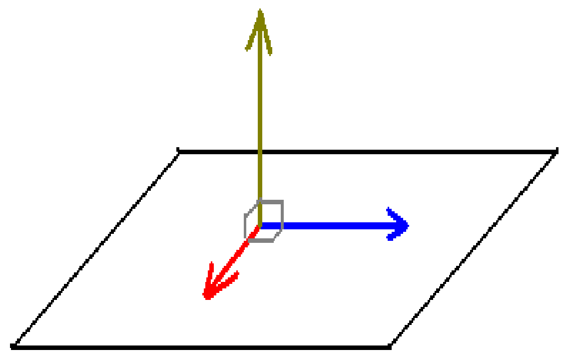
To get out of the plane, we must give our imaginary "car" the ability to fly. Suppose the car can move in two different ("fundamental") directions in the plane, as above. Suppose also you can steer it in a third direction which allows you to fly up out of the plane (which means that by putting it in reverse, you can burrow down below the plane also):



Suppose the blue and red directions in the plane are perpendicular and that the green direction is drawn at a $45°$ angle to the blue direction. This time we can reach any point in "space" (here, the term space is used, in contrast to line and plane, to refer to our usual "3-dimensional" Euclidean space). Let's refer to locations in this space in terms of left-right (the blue direction), forward-back (the red direction), and up-down (NOT the green direction!). If we wanted to start a journey and travel 10 units forward, 5 units to the right, and 10 units up, how would we do it? Well, forward and right movement is rather easy. The only way to get upward movement is to travel in the green direction. But this also introduces rightward movement — but we can remove any extra rightward movement with leftward movement (by driving in the blue direction in reverse). So let's move in the green direction $10\sqrt{2}$ units. This means that we have moved up out of the plane 10 units and also to the right 10 units (remember what Pythagoras says: $10^2 + 10^2 = (10\sqrt{2})^2$). Now we can "adjust" the left-right distance by traveling in reverse 5 units in the blue direction (i.e., units to the left). This only leaves the forward (red) direction to deal with and so we go 10 units in the red direction. In fact, we can describe the location of our destination as: $10\sqrt{2}$ green, -5 blue, 10 red (notice the use of a negative to indicate that we traveled in the blue direction in reverse).

Question. What combination of green, blue, red would move you x units forward, y units to the right, and z units up?

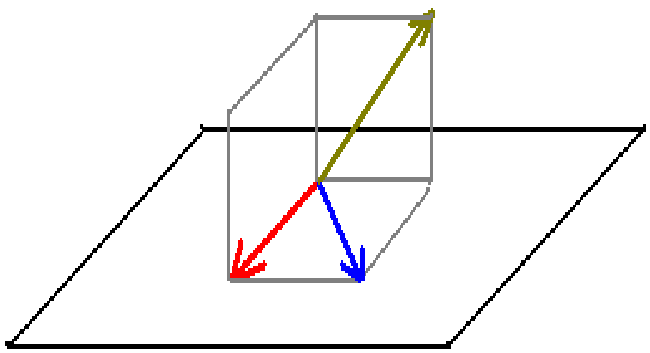
Of course the questions given above would be a bit easier if our fundamental directions were each perpendicular to the other (if they were *mutually orthogonal*). This can be accomplished by making the green direction straight up out of the plane (it can also be accomplished by making the green direction straight down):



Now it is easy to see how to go a certain distance forward, to the right, and up. To go x units forward, y units to the right, and z units up, simply go x units in the red direction, y units in the blue direction, and z units in the green direction. These travel plans are even valid if x, y and/or z are negative (just put the car in reverse, as needed). You might also notice that any other direction in this space can be thought of as a combination of these three fundamental directions. Therefore this space is 3-dimensional — we can travel to any point in this space provided we can travel in three fundamental directions. Here, by "fundamental direction" we mean three directions, no one of which is a combination of the other two (i.e., three *linearly independent* directions).

So this story of a flying car with a strange steering mechanism gives us some intuition as to why lines are 1-dimensional, planes are 2-dimensional, and space (as we are familiar with in our experience) is 3-dimensional. Now, we need to think very abstractly to move up another dimension. We want our imaginary car to have the ability to point in a new direction that is independent of left-right, forward-back, and up-down. Of course we cannot visualize such a direction since we live in the usual 3-dimensional space described above and we do not have physical access to a fourth fundamental direction (but we can access it mentally). Just as we moved from the line to the plane by introducing a new direction not on the line, and as we moved from the plane to the 3-dimensional space by introducing a new direction not in the plane, we now move from 3-dimensions to 4-dimensions by introducing a new direction not in our 3-D space. In fact, things seemed easier in the previous conversation when we had fundamental directions which were perpendicular to each other ("mutually orthogonal"), so let's start with the left-right, forward-back, up-down directions and add a new direction perpendicular to each of these. Since the 4-dimensional space we are moving into is often called *hyperspace*, let's call the new direction "hyperup" (when we go hyperup in reverse, we are moving "hyperdown"). Now if we want to go x units forward, y units to the left, z units up, and 0 units hyperup, then we just travel in the manner described above in our usual 3-D space. In this case, our 3-D space is a slice of 4-D hyperspace (the slice we get by going 0 units hyperup). In fact, if we go 1 unit hyperup and then "look around" in the other three fundamental directions, we find another copy of our usual 3-D space — it's just moved hyperup 1 unit from our familiar copy of 3-D space. In fact, just as you can think of a plane as being produced by taking a bunch of copies of a line, one on top of another (this is where the text "inks up" a line and drags it), and just as you can think of our 3-dimensional space as a bunch of planes lying one on top of another, we can think of 4-D hyperspace as a pile of copies of our usual 3-D space piled hyperup on top (or is it *hypertop*?) of each other. Of course, this is hard to visualize since we cannot see hyperup! Now to describe a destination in hyperspace, we must describe it in terms of movement (from our starting point) forward, right, up, and hyperup. In fact, any destination can be uniquely described in terms of x units forward, y units right, z units up, and w units hyperup, for some x, y, z, and w (where these may be negative). We could take some other collection of four directions to be our four fundamental directions (as long as they are "linearly independent"), but these four are the most convenient.

Question. What is another collection of four fundamental directions in hyperspace?
Answer. We *could* take (1) forward-back, (2) forward right-back left, (3) upper right-lower left, and (4) hyperup forward-hyperdown back:

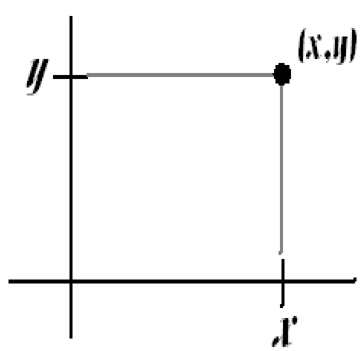


This is just one option — there are an infinite number of other possible choices.

So where is hyperspace? You could say that we are in hyperspace, we just are trapped in this 3-dimensional cross section (we cannot go hyperup or hyperdown… we are stuck at hyper-sealevel!). An analogy which we can visualize is the situation of the 2-dimensional flatlander that has no concept of "up." You might say that the flatlander is trapped in its 2-dimensional cross section of our 3-D space. To talk more quantitatively about hyperspace, let's look at coordinates.

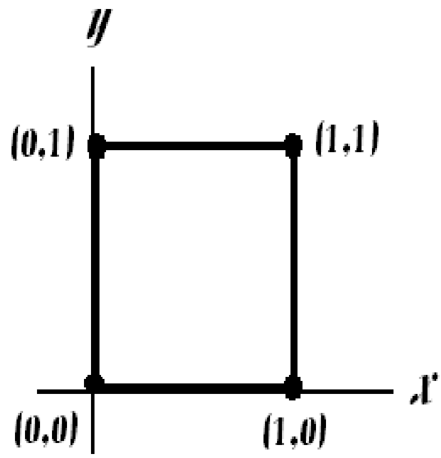
Coordinates and Dimension

We are familiar with describing the location of a point on a line (a 1-dimensional space) with a real number. In the plane, we locate a point by giving an x-coordinate and a y-coordinate:

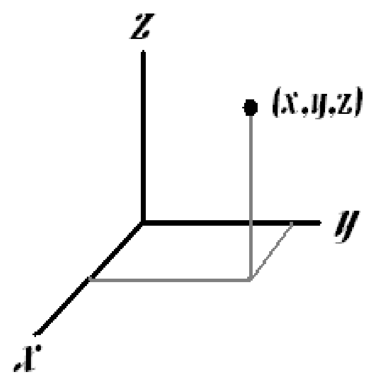


Notice that the x-axis and y-axis are perpendicular (like our left-right and up-down directions discussed in the previous section). As we saw in the past, the distance between points (x_1, y_1) and

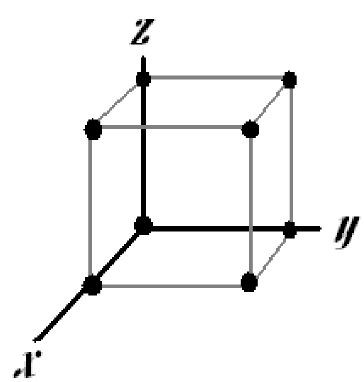
(x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (as given by the Pythagorean Theorem). We can give the coordinates of the corners of a square in the plane as $(0, 0)$, $(0, 1)$, $(1, 1)$:



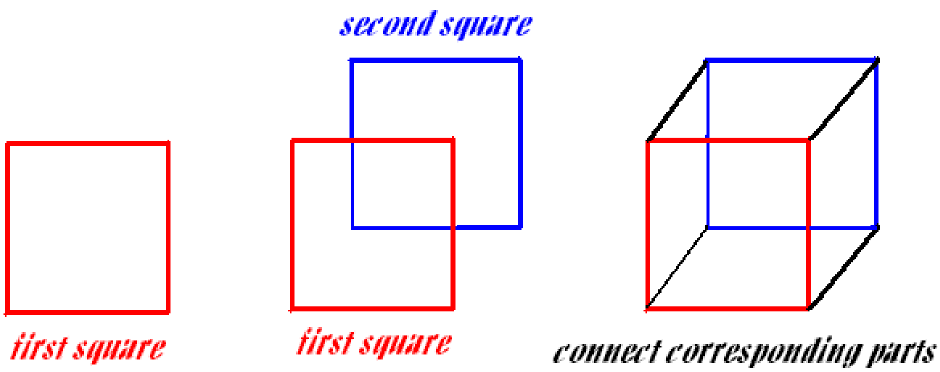
In 3-dimensions, we can locate a point by giving three coordinates, say (x, y, z):



Again, from the Pythagorean Theorem, the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $d\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. We can give the coordinates of the corners of a cube in 3-dimensions as $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$, $(1, 1, 1)$ (notice that this is all possible triples of 0's and 1's):

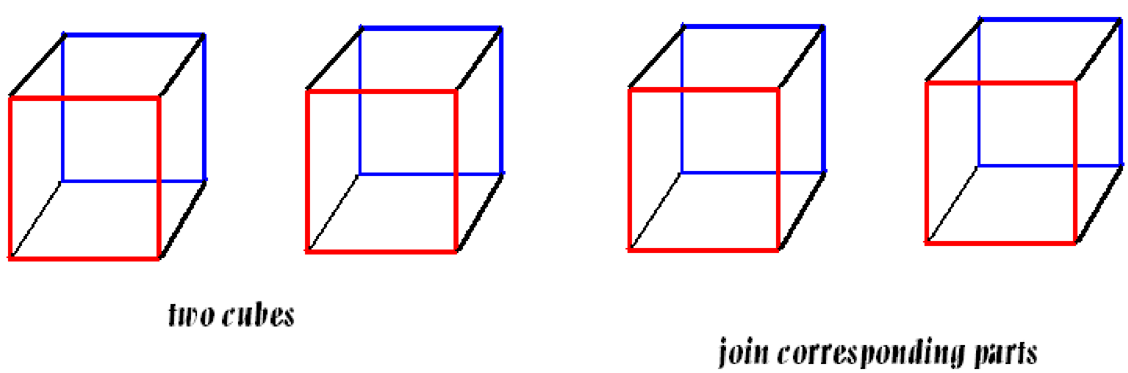


As with the square, two points are on the same edge when they differ in exactly one coordinate. Also, notice that what we really have in the cube is a copy of the square in the xy-plane and a copy of the square in the plane $z = 1$ (i.e., the plane parallel to the xy-plane which is 1 unit above the xy-plane), where corresponding corners are connected. In fact, that is how we usually draw a cube on paper. We draw a square, draw a second square displaced from the first, and then connect corresponding points:



In fact, the square can be thought of in the same way as two line segments displaced from each other with corresponding points (endpoints) connected. Notice, as mentioned in the text, that our pictures of cubes drawn on paper distort some angles and distances. This is the price of "projecting" a 3-dimensional cube onto a 2-dimensional plane (you can think of these drawings as shadows of a 3-D cube on a 2-D plane). We will pay the same price when we project a 4-D hypercube into a 3-D space.

Now let's consider 4-dimensional hyperspace. We do so simply by talking about points with four coordinates: (x, y, z, w). As discussed in the previous section, these coordinates represent movement forward, right, up, and hyperup. By analogy with the cube in 3-D, we can write the coordinates of the corners of the 4-D cube (called a *hypercube*) as the 16 ($16 = 2^4$) points with coordinates 0 and 1. Two corners are on the same edge if they differ in exactly one coordinate. We can think of this as one 3-D cube with hyperup coordinate 1. We then join the corresponding parts of the two 3-D cubes:



We measure distances in hyperspace in a way analogous to the lower dimensional space. The distance from (x_1, y_1, z_1, w_1) to (x_2, y_2, z_2, w_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2}$. We then see that in the hypercube (as with the 3-D cube) two corners are on the same edge if and only if they are 1 unit apart.

So with coordinate notation, we see that a point in an n-dimensional space can be represented as an n-tuple of numbers: $(x_1, x_2, \ldots, x_n)$. The distance between two such points $(x_1, x_2, \ldots, x_n)$ and

$(y_1, y_2, \ldots, y_n)$ is $d = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$. This idea is sometimes called the "Fundamental Theorem of Finite Dimensional Vector Spaces." A cube in this space has 2^n corners (consisting of the 2^n possible collections of 0's and 1's) with two corners on the same edge if they are distance 1 apart.