# 7.1 Chance Surprises, 7.2 Predicting the Future in an Uncertain World, 7.4 Down for the Count

Probability deals with predicting the outcome of future experiments in a quantitative way. The experiments must be easily quantifiable and the possible outcomes well defined. The *sample space* of an experiment is a list of the possible outcomes of the experiment. If there are several outcomes, say $n$ of them, each of equal likelihood, then the *probability* of a particular outcome is $1/n$.

**Question.** Consider the experiment of rolling a 6-sided (cubical) die. What is the probability of rolling a 4?
**Answer.** Well, there are 6 possible outcomes, namely rolling a 1, 2, 3, 4, 5, or 6. Each is equally likely (assuming a fair die), and so the probability of any of these events is 1/6. In particular, the probability of rolling a 4 is 1/6.

In general, for an experiment with a finite number of possible outcomes, the probability of event $E$, denoted $P(E)$, is

$$P(E) = \frac{N}{T} = \frac{\#\text{ of possible ways } E \text{ can occur}}{\#\text{ of total possible outcomes}}.$$

**Question.** Roll a 6-sided die. What is the probability of getting an even number? An odd number? A prime number?

**Question.** Roll two 6-sided dice. What is the sample space? What is the probability of each outcome? (HINT: There are two dice so there is a first die and a second die — you could color them red and blue.) What are the possible sums on the die and what are their probabilities?
**Answer.** We list the outcome of the pair of dice as an ordered pair (with the result of the red die listed first, say). We then get:

| | | | | | |
|---|---|---|---|---|---|
| (1,1)=2 | (1,2)=3 | (1,3)=4 | (1,4)=5 | (1,5)=6 | (1,6)=7 |
| (2,1)=3 | (2,2)=4 | (2,3)=5 | (2,4)=6 | (2,5)=7 | (2,6)=8 |
| (3,1)=4 | (3,2)=5 | (3,3)=6 | (3,4)=7 | (3,5)=8 | (3,6)=9 |
| (4,1)=5 | (4,2)=6 | (4,3)=7 | (4,4)=8 | (4,5)=9 | (4,6)=10 |
| (5,1)=6 | (5,2)=7 | (5,3)=8 | (5,4)=9 | (5,5)=10 | (5,6)=11 |
| (6,1)=7 | (6,2)=8 | (6,3)=9 | (6,4)=10 | (6,5)=11 | (6,6)=12 |

There are 36 possible outcomes listed here, each one equally likely, so the probability of each is 1/36. The sums can be 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12. The probabilities of these events are: $P(2) = 1/36$, $P(3) = 2/36$, $P(4) = 3/36$, $P(5) = 4/36$, $P(6) = 5/36$, $P(7) = 6/36$, $P(8) = 5/36$, $P(9) = 4/36$, $P(10) = 3/36$, $P(11) = 2/36$, and $P(12) = 1/36$.

We can actually perform a particular experiment a number of times and record the frequency of various outcomes. The *relative frequency* of an event is the number of times it occurs divided by the number of times the experiment was performed.

**Example.** Roll a 6-sided die 6 times and record the relative frequency of each outcome.

**Example.** Roll a 6-sided die 36 times and record the relative frequency of each outcome.

We *should* observe that the second of the above two trials yields results closer to our expectations. This is explained by the following.

**The Law of Large Numbers.** If an experiment is repeated a large number of times, then the relative frequency of a particular outcome will tend to be close to the probability of that particular outcome.

As a corollary to the Law of Large Numbers, we can conclude that improbable events are likely to occur if an experiment is performed enough times. This can explain, in part, amazing coincidences. A particular event may be very improbable, but after a large number of trials, we may see amazing outcomes. In fact, unlikely events are inevitable.

We have to be careful in how we interprete probability. We can easily see a 1-in-a-million event. We merely need to perform an experiment with one million equally likely possible outcomes and then perform the experiment. This can be done by flipping a coin a certain number of times. Each time we flip the coin, we get a head or tail with equal probability (namely $1/2$). If we flip a coin 20 times, then there are $2^{20} = 1,048,576$ possible outcomes, each equally likely. I cannot tell you *up front* what the outcome will be. However, once the experiment is performed then, whatever the outcome, it had probability 1/1,048,576. Let's perform the experiment.

A fundamental property of probability is that probabilities are calculated in terms of information available. This property leads to the very foundation of statistics: conditional (or Bayesian) probability.

**Question.** Consider a deck of 5 cards consisting of an ace of spaces ($A_1$), ace of clubs ($A_2$), a king ($K$), a queen ($Q$), and a jack ($J$). Suppose you are given two cards at random. What is the sample space and what is the probability that you get two aces?

**Question.** In the above experiment, suppose you know that you have been given an ace. With this information, what is the probability you have 2 aces? NOTE: An erroneous argument is this: "Well, I have one ace, so that leaves 4 other cards. One of them is an ace, so the probability I have that other ace as well is $1/4$."

**Question.** Suppose you roll a 6-sided die. What is the probability of rolling a 4? What is the probability of rolling a 4 IF you know that you have rolled an even number? What is the probability of rolling a 4 given that you have rolled an integer power of 2? What is the probability of rolling a 2 given that you have rolled an integer power of 3?

The probability of two events happening sequentially (when the events do not depend on each other — they are independent) is the *product* of the events.

**Question.** What is the probability of getting two consecutive 6's on a 6-sided die?

**Question.** What is the probability of getting 4 heads on four coin tosses? What's the probability of getting all heads on 20 coin tosses? What is the probability of getting HTHHHTTHTHHHTTT-THHTH on 20 coin tosses? If you toss a coin an infinite number of times, what's the probability of getting all heads?

The sum of the probabilities of all of the events in a probability space is 1. So the probability of an event $E$ plus the probability that $E$ does not occur (denoted $P(\tilde{E})$) is: $P(E) + P(\tilde{E}) = 1$. A useful notation in the computation of certain probabilities, is the "factorial" notation. For a counting number $n$ define $n$ *factorial* as $n! = 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n$.

**Example.** We have $1! = 1$, $2! = 1 \times 2 = 2$, $3! = 1 \times 2 \times 3 = 6$, $4! = 1 \times 2 \times 3 \times 4 = 24$. Notice that $5!$ is just 5 times $4!$, so $5! = 5 \times 4! = 5 \times 24 = 120$. Similarly, $6! = 720$ and $7! = 5040$. In fact, we can use factorials to get products of consecutive counting numbers, even if we don't start at 1. For example, we can write $10 \times 9 \times 8$ as $10!/7!$.

**Question.** In the following conversation, pretend that there are exactly 365 possible birthdays. What is the probability that a person has a birthday? (Answer: $365/365 = 1$.) What is the probability that 2 people have different birthdays? What is the probability that 3 people have different birthdays? What is the probability that $n$ people all have different birthdays (express using factorials)?

**Question.** A more interesting question than the above is: What is the probability that in a group of $n$ people, there are at least two people that share a birthday? This is the negation of the previous problem, so the answer for this question is 1 minus the answer to the previous question. The results are surprising! From Section 7.2, we have:

| **Number of People** ($n \leq 365$) | **Probability of at least Two Sharing a Birthday** |
|---|---|
| 5 | 0.027 |
| 10 | 0.116 |
| 15 | 0.252 |
| 20 | 0.411 |
| 23 | 0.501 |
| 25 | 0.568 |
| 30 | 0.706 |
| 40 | 0.891 |
| 50 | 0.970 |
| 60 | 0.994 |
| 70 | 0.9991 |
| 80 | 0.99991 |
| 90 | 0.999993 |
| $n$ | $1 - \frac{365!}{(365-n)!365^n}$ |

We can use factorials to count the number of ways of arranging things. For example, suppose you want to count the number of ways a race can end in which there are 6 runners. There are 6 possible runners which could finish first. This leaves 5 runners, any one of which could finish second. There are 4 possible runners to finish third, 3 to finish fourth, 2 to finish fifth, and 1 to finish last. So the total number of possible outcomes to such a race is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$.

**Question.** A standard deck of cards consists of 52 cards. If you perform the experiment of laying 5 cards out on a table *in order*, then how many possible outcomes are there for this experiment?

The previous example may not reflect exactly what we are interested in when we are counting hands of cards (or, as we will see, choosing lottery numbers). Often we are interested in *which* cards we have, and not in *what order* the cards were received. For example, we don't distinguish between the hand $(A\spadesuit, J\clubsuit, 2\heartsuit, 7\diamondsuit, K\clubsuit)$ and the hand $(J\clubsuit, K\clubsuit, 2\heartsuit, A\spadesuit, 7\diamondsuit)$. Both hands contain the same cards, just in different orders. Since there are 5 cards in the hand, there are $5! = 120$ different ways to arrange them. So if we want to count the possible number of 5 card hands (ignoring the order in which the cards were received), then we get the number of ways the cards could have been received ($52 \times 51 \times 50 \times 49 \times 48 = 311,875,200$, the answer to the previous example) divided by the total number of ways to arrange the cards ($5! = 120$). So there are a total of $\dfrac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$ possible hands of cards. In general, if we want to count the number of ways to *arrange* $m$ items chosen from a set of size $n$, the we are interested in the number of *permutations of* $m$ *things from a set of size* $n$ and this quantity is $P(n, m) = \dfrac{n!}{(n-m)!}$. Notice that it is **order** that matters in permutations. If we only want to count the number of ways to *have* $m$ items from a set of size $n$, then we are interested in the number of *combinations of* $m$ *things from a set of size* $n$ and this quantity is $C(m,n) = \dfrac{n!}{(n-m)!m!}$. Notice that **order does not matter** in combinations.

**Question.** (**Burger King** problem from Section 7.4.) You take a summer job making hamburgers. The burgers can be made with any of the following: cheese, lettuce, tomato, pickles, onions, mayo, catsup, and mustard. How many different kinds of burgers can you make?

**Question.** In the Virginia State Lottery game "Cash 5," you pick 5 numbers from the counting numbers 1 through 34. The lottery commission then draws 5 numbers and you want to match their numbers. On a \$1.00 ticket, you win \$100,000 if you match all 5 of their numbers, \$100 if you match 4 numbers, and \$5 if you match 3 numbers. What is the probability of each of these events? Do you want to play this game many, many times? We will explore this example again when we look into expected value.