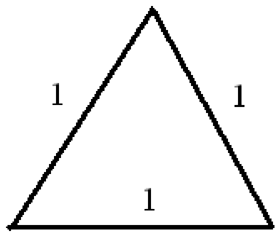
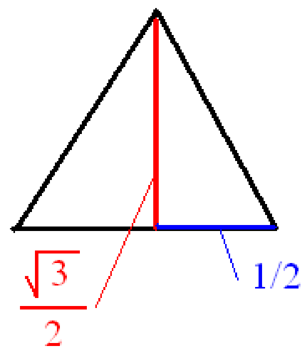
4.5 The Platonic Solids Turn Amorous, "Self Duals" Problem

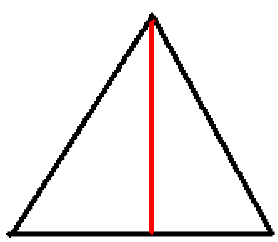
Recall that the center of a triangle is determined by finding the intersection of the bisectors of each angle. Since the faces of a tetrahedron are equilateral triangles, we want to find the location of the center of an equilateral triangle with edge length 1. First, cut such a triangle in half and notice the resulting lengths (determined, in part, from the Pythagorean Theorem):



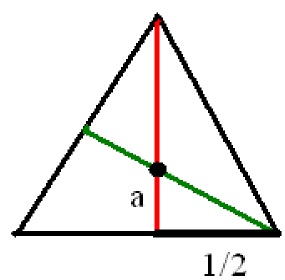
Equilateral Triangle



The two halves are 30-60-90 triangles. Next, bisect one of the other angles of the equilateral triangle. We want to find the point of intersection of these two bisectors.

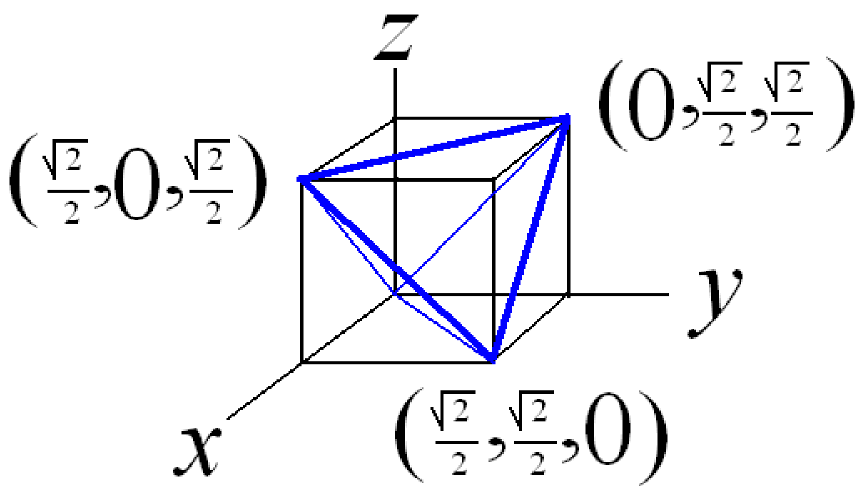


Equilateral Triangle

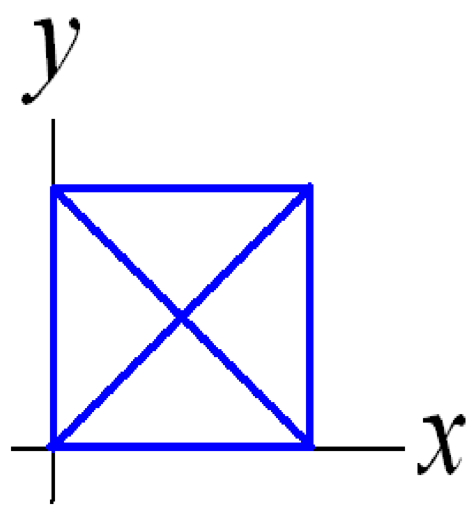


We see that this produces a smaller 30-60-90 triangle with one side of length 1/2 and another side we label as length a. By similar triangles, we know that $\frac{a}{1/2} = \frac{1/2}{\sqrt{3}/2}$ or $a = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$. So the center of an equilateral triangle with sides of length 1 is located $\sqrt{3}/6$ units from each edge.

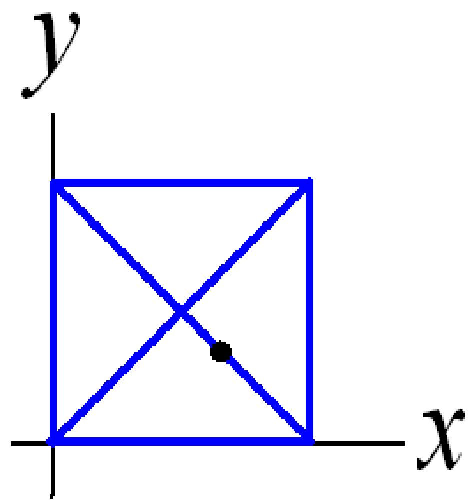
Now for the tetrahedron, let's consider coordinates and a cube with sides of length $\sqrt{2}/2$. Then we can consider a tetrahedron which shares vertices with the cube and has edges of length 1 (that is how we were able to put our model tetrahedron inside our model cube):



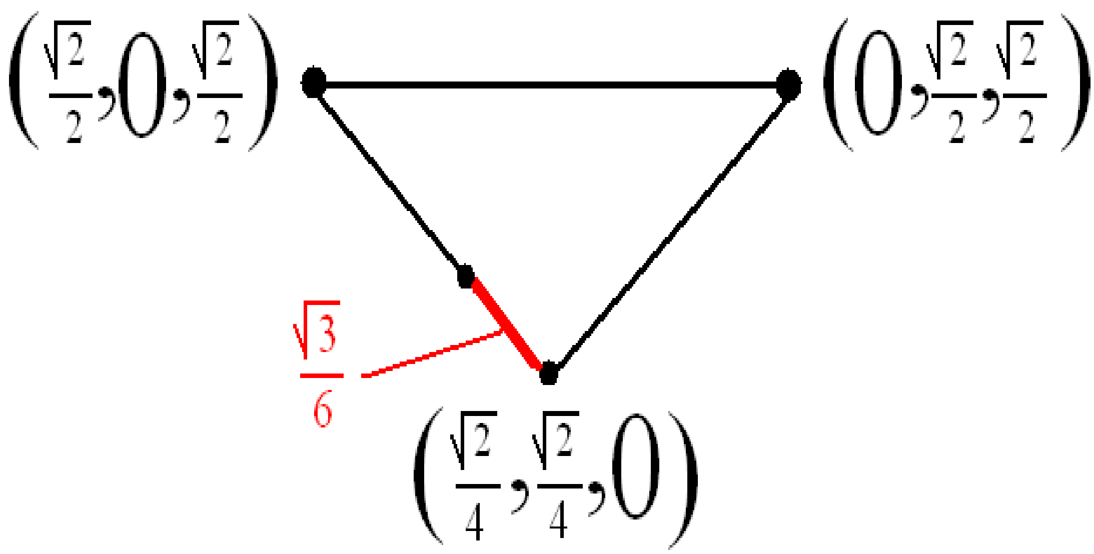
Now the projection ("shadow") of this tetrahedron is the xy-plane is



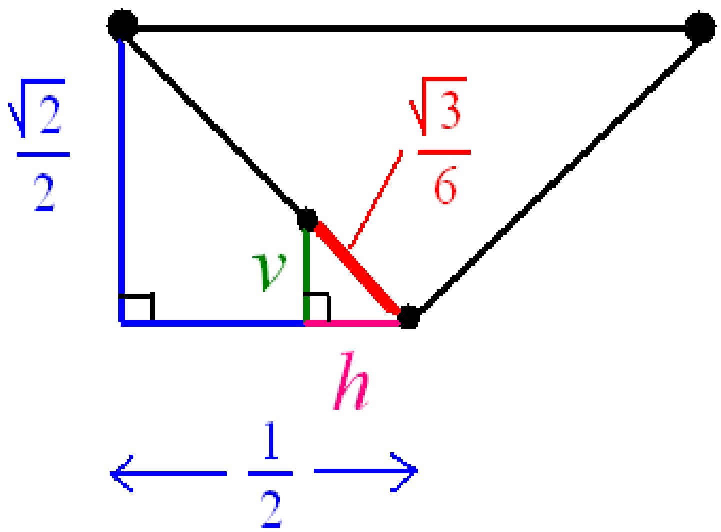
Consider the face of the tetrahedron through points $(0,0,0)$, $(\sqrt{2}/2, 0, \sqrt{2}/2)$, and $(\sqrt{2}/2, \sqrt{2}/2, 0)$. We find the center by first going half way across the base to the point $(\sqrt{2}/4, \sqrt{2}/4, 0)$. Next, we go a distance $\sqrt{3}/6$ towards the vertex at $(\sqrt{2}/2, 0, \sqrt{2}/2)$. We need to find the coordinates where this leaves us. In the xy-plane we see the point as:



Let's cut the tetrahedron by a vertical plane through $(\sqrt{2}/2, 0, \sqrt{2}/2)$ and $(0, \sqrt{2}/2, \sqrt{2}/2)$. This plane contains the desired point:



We introduce two new triangles and label known and unknown distances:

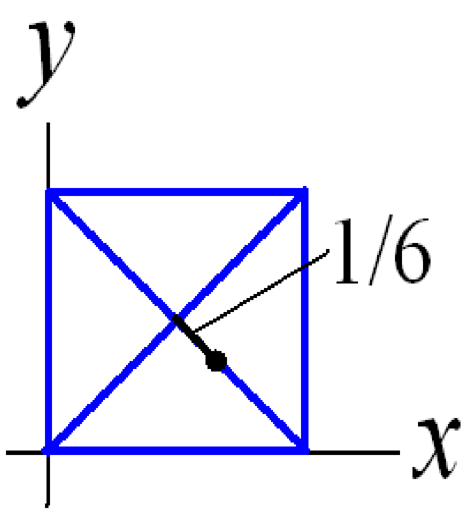


By similar triangles, we know $\dfrac{v}{h} = \dfrac{\sqrt{2}/2}{1/2} = \sqrt{2}$. So $v = \sqrt{2}h$. By the Pythagorean Theorem we know:

$$v^2 + h^2 = \left(\frac{\sqrt{3}}{6}\right)^2 \text{ or } (\sqrt{2}h)^2 + h^2 = \frac{3}{36}$$

$$\text{or } 3h^2 = \frac{3}{36} \text{ or } h = \frac{1}{6} \text{ and } v = \frac{\sqrt{2}}{6}.$$

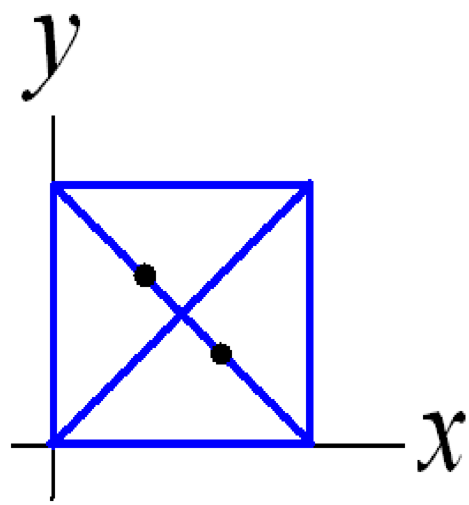
So in the xy-plane, the distance from $(\sqrt{2}/4, \sqrt{2}/4, 0)$ to the shadow of the desired point is $1/6$:



We find that the coordinates of the desired point are

$$(x, y, z) = \left(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{6}\right) = \left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}\right).$$

A similar analysis of the face passing through $(0,0,0)$, $(\sqrt{2}/2, \sqrt{2}/2, 0)$, and $(0, \sqrt{2}/2, \sqrt{2}/2)$ yields the coordinates of the center of this face as $\left(\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{6}\right)$. In the xy-plane, we get the projection of the tetrahedron and these two points as:



So the distance between these two points is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. THIS is the answer to the question asked.

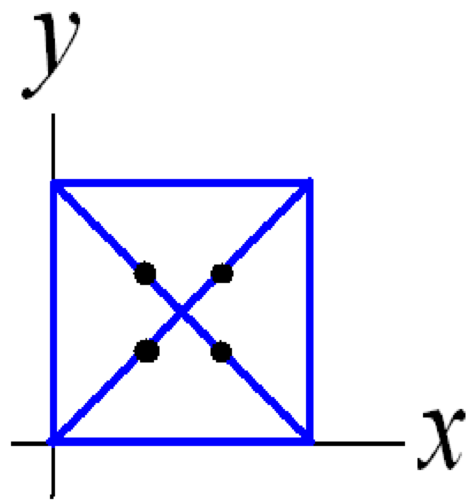
Notice that we can calculate the distance directly as

$$\sqrt{\left(\frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{6}\right)^2 + \left(\frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6}\right)^2} = \sqrt{\left(\frac{\sqrt{2}}{6}\right)^2 + \left(\frac{\sqrt{2}}{6}\right)^2} = \sqrt{\frac{4}{36}} = \frac{2}{6} = \frac{1}{3}.$$

A similar argument gives the other two centers as

$$\left(\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{3}\right) \text{ and } \left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right)$$

with the projections onto the xy-plane of all four centers as



You can verify that each pair of these points is distance 1/3 apart.