Chapter 2. Vectors and Vector Spaces  
2.1. Operations on Vectors—Proofs of Theorems

**Theorem 2.1.1(A1).** Properties of Vector Algebra in \( \mathbb{R}^n \).

Let \( x, y, z \in \mathbb{R}^n \). Then:

**A1.** \((x + y) + z = x + (y + z)\) (Associativity of Vector Addition)

**Proof.** Let \( x, y, z \in \mathbb{R}^n \) be \( x = [x_1, x_2, \ldots, x_n] \), \( y = [y_1, y_2, \ldots, y_n] \), and \( z = [z_1, z_2, \ldots, z_n] \). Then:

\[
(x + y) + z = ([x_1, x_2, \ldots, x_n] + [y_1, y_2, \ldots, y_n] + [z_1, z_2, \ldots, z_n] \\
= [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n] + [z_1, z_2, \ldots, z_n] \\
= [x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots, x_n + y_n + z_n] \\
= [x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots, x_n + (y_n + z_n)] \\
= x + (y + z).
\]

Theorem 2.1.1(A1) (continued)

Theorem 2.1.1. Properties of Vector Algebra in \( \mathbb{R}^n \).

Let \( x, y, z \in \mathbb{R}^n \). Then:

**A1.** \((x + y) + z = x + (y + z)\) (Associativity of Vector Addition)

**Proof (continued).**

\[
(x + y) + z = [x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots, x_n + (y_n + z_n)] \\
\text{since addition in } \mathbb{R} \text{ is associative} \\
= [x_1, x_2, \ldots, x_n] + [y_1, z_1, y_2, z_2, \ldots, y_n, z_n] \\
\text{by the definition of vector addition} \\
= [x_1, x_2, \ldots, x_n] + ([y_1, y_2, \ldots, y_n] + [z_1, z_2, \ldots, z_n]) \\
\text{by the definition of vector addition} \\
= x + (y + z).
\]

**Theorem 2.1.2.** Let \( V_1 \) and \( V_2 \) be vector spaces of \( n \)-vectors. Then \( V_1 \cap V_2 \) is a vector space.

**Proof.** By our definition of “vector space,” we only need to prove that \( V_1 \cap V_2 \) is closed under linear combinations. Let \( x, y \in V_1 \cap V_2 \) and \( a, b \in \mathbb{R} \). Since \( V_1 \) is a vector space then it is closed under linear combinations and so \( ax + by \in V_1 \). Similarly, \( ax + by \in V_2 \). So \( ax + by \in V_1 \cap V_2 \). Since \( x \) and \( y \) are arbitrary elements of \( V_1 \cap V_2 \) and \( a, b \in \mathbb{R} \) are arbitrary scalars, then \( V_1 \cap V_2 \) is closed under linear combinations. That is, \( V_1 \cap V_2 \) is a vector space.
Theorem 2.1.3. If $V_1$ and $V_2$ are vector spaces of $n$-vectors, then $V_1 + V_2$ is a vector space.

Proof. By our definition of “vector space,” we must show that $V_1 + V_2$ is closed under linear combinations. Let $x, y \in V_1 + V_2$ and let $a, b \in \mathbb{R}$. Then $x = x_1 + x_2$ and $y = y_1 + y_2$ for some $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$. Since $V_1$ and $V_2$ are vector spaces, then they are closed under addition and so $ax_1 + by_1 \in V_1$ and $ax_2 + by_2 \in V_2$. Therefore

$$ax + by = a(x_1 + x_2) + b(y_1 + y_2) = (ax_1 + by_1) + (ax_2 + by_2) \in V_1 + V_2.$$ 

Since $x$ and $y$ are arbitrary vectors in $V_1 + V_2$ and $a, b \in \mathbb{R}$ are arbitrary scalars, then we have that $V_1 + V_2$ is closed under linear combinations. That is, $V_1 + V_2$ is a vector space.

Theorem 2.1.4. If vector spaces $V_1$ and $V_2$ are essentially disjoint then every element of $V_1 \oplus V_2$ can be written as $v_1 + v_2$, where $v_1 \in V_1$ and $v_2 \in V_2$, in a unique way.

Proof. Let $V_1$ and $V_2$ be essentially disjoint vector spaces of $n$-vectors; that is, $V_1 \cap V_2 = \{0\}$. Suppose some $v \in V_1 \oplus V_2$ is of the form $v = v_1 + v_2 = v'_1 + v'_2$ where $v_1, v'_1 \in V_1$ and $v_2, v'_2 \in V_2$. Then $v_1 - v'_1 = v'_2 - v_2$. So $v_1 - v'_1 \in V_1$ and $v'_2 - v_2 \in V_2$ since $V_1$ and $V_2$ are vector spaces. But then $v_1 - v'_1, v'_2 - v_2 \in V_1 \cap V_2$ and so $v_1 - v'_1 = 0$ and $v'_2 - v_2 = 0$. That is, $v_1 = v'_1$ and $v'_2 = v_2$. So the representation of $v \in V_1 \oplus V_2$ as a sum of an element of $V_1$ and an element of $V_2$ is unique, as claimed.

Theorem 2.1.5. If $\{v_1, v_2, \ldots, v_k\}$ is a basis for a vector space $V$, then each element can be uniquely represented as a linear combination of the basis vectors.

Proof. Suppose that

$$x = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k.$$ 

Then

$$0 = x - x = (b_1 v_1 + b_2 v_2 + \cdots + b_k v_k) - (c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = (b_1 - c_1) v_1 + (b_2 - c_2) v_2 + \cdots + (b_k - c_k) v_k.$$ 

Since the basis consists (by definition) of a linearly independent set of vectors, then $b_1 - c_1 = b_2 - c_2 = \cdots = b_k - c_k = 0$; that is, $b_1 = c_1, b_2 = c_2, \ldots, b_k = c_k$. Therefore, the representation of $x$ is unique. Since $x$ is an arbitrary vector in $V$, the claim follows.

Theorem 2.1.6(3). Properties of Inner Products.

Let $x, y \in \mathbb{R}^n$ and let $a \in \mathbb{R}$. Then:

3. $a(x, y) = \langle ax, y \rangle$ (Factoring of Scalar Multiplication in Inner Products).

Proof. Let $x, y \in \mathbb{R}^n$ be $x = [x_1, x_2, \ldots, x_n]$ and $y = [y_1, y_2, \ldots, y_n]$. Then

$$a(x, y) = a([x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_n]) = a(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n) \text{ by the definition of } \langle x, y \rangle = a(x_1 y_1) + a(x_2 y_2) + \cdots + a(x_n y_n) \text{ by distribution property of multiplication over addition in } \mathbb{R} = \left( a(x_1) y_1 + (ax_2) y_2 + \cdots + (ax_n) y_n \right) \text{ by associativity for multiplication in } \mathbb{R} = \langle ax_1, ax_2, \ldots, ax_n \rangle, [y_1, y_2, \ldots, y_n] \text{ by the definition of inner product} = \langle ax, y \rangle.$$


Theorem 2.1.7. Schwarz Inequality.
For any \( x, y \in \mathbb{R}^n \) we have \( |\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \).

Proof. Let \( t \in \mathbb{R} \). Then

\[
0 \leq \langle (tx + y), (tx + y) \rangle \text{ by Theorem 2.1.6(1)} \\
= t\langle x, tx + y \rangle + \langle y, tx + y \rangle \text{ by linearity in the 1st entry} \\
= t\langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + t\langle y, y \rangle \text{ by linearity in the 2nd entry} \\
= at^2 + bt + c \text{ by Theorem 1.2.6(2)}
\]

where \( a = \langle x, x \rangle, b = 2\langle x, y \rangle, \) and \( c = \langle y, y \rangle \). As a quadratic in \( t \),

\( at^2 + bt + c \) cannot have two distinct roots or else we would have

\( at^2 + bt + c < 0 \) for some \( t \). This means that the discriminant \( b^2 - 4ac \) in

the quadratic equation \( t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), must be

\( b^2 - 4ac \leq 0 \); that is, \( (b/2)^2 \leq ac \). Hence, we have \( (b/2)^2 = \langle x, y \rangle^2 \leq ac = \langle x, x \rangle \langle y, y \rangle \) or \( \sqrt{\langle x, x \rangle \langle y, y \rangle} \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \). \( \square \)

Theorem 2.1.8 (continued 1)

Proof (continued). Finally,

\[
\rho(x + y)^2 = \rho((c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_k + d_k)v_k)^2 \\
= \sum_{j=1}^k (c_j + d_j)^2 = \sum_{j=1}^k (c_j^2 + 2c_jd_j + d_j^2) \\
= \sum_{j=1}^k c_j^2 + 2\sum_{j=1}^k c_jd_j + \sum_{j=1}^k d_j^2 \\
\leq \sum_{j=1}^k c_j^2 + 2\left( \sum_{j=1}^k c_j^2 \right)^{1/2} \left( \sum_{j=1}^k d_j^2 \right)^{1/2} + \sum_{j=1}^k d_j^2 \\
\leq \left( \sum_{j=1}^k c_j^2 \right)^{1/2} + \left( \sum_{j=1}^k d_j^2 \right)^{1/2} \quad \text{by Theorem 2.1.7 (Schwarz's Inequality in } \mathbb{R}^n) \\
= \left( \sum_{j=1}^k c_j^2 \right)^{1/2} + \left( \sum_{j=1}^k d_j^2 \right)^{1/2} \quad \text{by Theorem 1.2.6(2)} \\
= (\rho(x) + \rho(y))^2.
\]

Theorem 2.1.8 (continued 2)

Proof (continued). . .

\( \rho(x + y)^2 - (\rho(x) + \rho(y))^2 \).

Taking square roots, \( \rho(x + y) \leq \rho(x) + \rho(y) \) and so the Triangle Inequality holds. Therefore \( \rho \) is a metric on \( V \). \( \square \)
**Theorem 2.1.10.** Every norm on (finite dimensional vector space) $V$ is equivalent to the basis norm $\rho$ for some given basis $\{v_1, v_2, \ldots, v_k\}$. Therefore, any two norms on $V$ are equivalent.

**Proof.** Let $\| \cdot \|_a$ be any norm on vector space $V$ and let $\{v_1, v_2, \ldots, v_k\}$ be a basis for the space. Then for some unique scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$ we have $x = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$. Then, by the Triangle Inequality and “Relation of Scalar Multiplication to Real Multiplication,”

$$\|x\|_a \leq \left( \sum_{i=1}^{k} |c_i|\|v_i\|_a \right) \leq \left( \sum_{i=1}^{k} |c_i|^2 \right)^{1/2} \left( \sum_{i=1}^{k} \|v_i\|_a^2 \right)^{1/2}.$$

Now with $[|c_1|, |c_2|, \ldots, |c_k|], [\|v_1\|_a, \|v_2\|_a, \ldots, \|v_k\|_a] \in \mathbb{R}^k$ we have by the Schwarz Inequality (Theorem 2.1.7) that

$$\sum_{i=1}^{k} |c_i|\|v_i\|_a \leq \left( \sum_{i=1}^{k} |c_i|^2 \right)^{1/2} \left( \sum_{i=1}^{k} \|v_i\|_a^2 \right)^{1/2}.$$

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**Theorem 2.1.10 (continued 2)**

**Proof (continued).** Define $f : C \rightarrow \mathbb{R}$ as $f(u) = \left\| \sum_{i=1}^{k} u_i v_i \right\|_a$. Gentle claims that $f$ is continuous (page 20). Let’s prove this. Let $y = \sum_{i=1}^{k} u_i v_i \in C$ and let $\varepsilon > 0$. Set $\delta = \varepsilon$. For any $\lambda - \sum_{i=1}^{k} u_i v_i \in C$ with $\|y - x\|_a < \delta$ we have

$$\varepsilon = \delta > \left\{ \begin{array}{l} \|y\|_a - \|x\|_a \\
\|x\|_a - \|y\|_a \\
\|y - x\|_a \end{array} \right\} = \left\{ \begin{array}{l} \|y\|_a - \|x\|_a \\
\|x\|_a - \|y\|_a \\
\|y - x\|_a \end{array} \right\} = \left\{ \begin{array}{l} f(y) - f(x) \\
f(x) - f(y) \end{array} \right\} \text{ if } f(y) \geq f(x) \\
f(x) - f(y) \text{ if } f(y) < f(x).$$

That is, $|f(y) - f(x)| < \varepsilon$. So $f : C \rightarrow \mathbb{R}$ is continuous.

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**Theorem 2.1.10 (continued 3)**

**Proof (continued).** By the Heine-Borel Theorem (since $C$ is closed and bounded and $V$ is finite dimensional), $C$ is compact and so continuous function $f$ attains a minimum value on $C$, say $f(u_*) \leq f(u)$ for all $u \in C$. Let $\bar{f} = f(u_*) > 0$. If $x = \sum_{i=1}^{k} u_i v_i \neq 0$ then

$$\|x\|_a = \left\| \sum_{i=1}^{k} c_i v_i \right\|_a = \left\{ \sum_{j=1}^{k} \left\| \sum_{i=1}^{k} u_i v_i \right\|_a^2 \right\}^{1/2} \left\{ \sum_{j=1}^{k} c_i^2 \right\}^{1/2} v_i = \rho(x)f(\bar{c})$$

where $\bar{c} = \sum_{i=1}^{k} \left( \frac{c_i}{\sum_{j=1}^{k} c_j^2} \right)^{1/2} v_i$, so $\bar{c} \in C$ since

$$\rho(\bar{c}) = \sum_{i=1}^{k} \left( \frac{c_i}{\sum_{j=1}^{k} c_j^2} \right)^{1/2} = \frac{1}{\sum_{j=1}^{k} c_j^2} \sum_{i=1}^{k} c_i^2 = 1.$$
Theorem 2.1.10. Every norm on (finite dimensional vector space) V is equivalent to the basis norm $\| \cdot \|_a$ for some given basis $\{v_1, v_2, \ldots, v_k\}$. Therefore, any two norms on V are equivalent.

Proof (continued). Since $F(x) \geq \| \cdot \|_a$ for all $x \in V$, $x \neq 0$. Of course $\|x\|_a \geq \|x\| = \|x\|_a \leq \exists \rho(x)$. That is, $\| \cdot \|_a \approx \rho(x)$. Since $\approx$ is an equivalence relation for Theorem 2.1.9, we have that any two norms on V are equivalent.

Theorem 2.1.11. A set of nonzero vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space with an inner product for which $\langle v_i, v_j \rangle = 0$ for $i \neq j$ (the vectors are said to be mutually orthogonal) is a linearly independent set.

Proof. Let $\{v_1, v_2, \ldots, v_k\}$ be a set of mutually orthogonal nonzero vectors. Assume the set is not linearly independent. Then $a_1 v_1 + a_2 v_2 + \cdots + a_i v_i + \cdots + a_k v_k = 0$ is satisfied where some coefficient is nonzero, say $a_i \neq 0$. So

$$v_i = (-a_1/a_i)v_1 + (-a_2/a_i)v_2 + \cdots + (-a_{i-1}/a_i)v_{i-1} + (-a_{i+1}/a_i)v_{i+1} + \cdots + (-a_k/a_i)v_k.$$

But then

$$\langle v_i, v_i \rangle = \langle (-a_1/a_i)v_1 + (-a_2/a_i)v_2 + \cdots + (-a_{i-1}/a_i)v_{i-1} + (-a_{i+1}/a_i)v_{i+1} + \cdots + (-a_k/a_i)v_k, v_i \rangle = 0,$$

a CONTRADICTION to the fact that $v_i \neq 0$. So the assumption that the set is not linearly independent is false; that is, the set is linearly independent, as claimed.