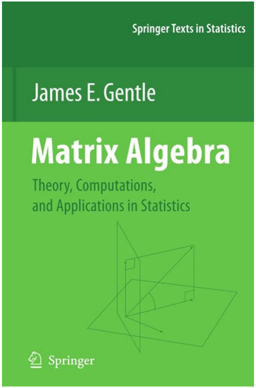
Theory of Matrices

Chapter 2. Vectors and Vector Spaces

2.2. Cartesian Coordinates and Geometrical Properties of Vectors
—Proofs of Theorems



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Theorem 2.2.1

Theorem 2.2.1. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for vector space V of n-vectors where the basis vectors are mutually orthogonal. Then $x \in V$ we have

$$x = \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle x, v_2\rangle}{\langle v_2, v_2\rangle} v_2 + \cdots \frac{\langle x, v_k\rangle}{\langle v_k, v_k\rangle} v_k.$$

Proof. Suppose $\{v_1, v_2, \ldots, v_k\}$ is a basis for V, then $x = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ for some (unique) scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

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$$\begin{aligned} \langle x, v_i\rangle &= \langle c_1 v_1 + c_2 v_2 + \cdots + c_k v_k, v_i\rangle \\ &= c_1\langle v_1, v_i\rangle + c_2\langle v_2, v_i\rangle + \cdots + c_i\langle v_i, v_i\rangle + \cdots + c_k\langle v_k, v_i\rangle \\ &= c_i\langle v_i, v_i\rangle \text{ since the basis is an orthogonal set.} \end{aligned}$$

Hence, $c_i = \dfrac{\langle x, v_i\rangle}{\langle v_i, v_i\rangle}$ and the claim follows. □

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Theorem 2.2.2

Theorem 2.2.2. The solutions to a homogeneous system of equations form a subspace of $\mathbb{R}^n$.

Proof. We need only show that the set of solutions to a homogeneous system is closed under linear combinations. Let x_1 and x_2 be solutions and let $a, b \in \mathbb{R}$ be scalars. Then

$$\begin{array}{cc} c_1^T x_1 = 0 & c_1^T x_2 = 0 \\ c_2^T x_1 = 0 & c_2^T x_2 = 0 \\ \vdots & \vdots \\ c_m^T x_1 = 0 & c_m^T x_2 = 0. \end{array}$$

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Hence

$$\begin{array}{rcl} c_1^T(ax_1 + bx_2) & = & ac_1^T x_1 + bc_1^T x_2 = a(0) + b(0) = 0 \\ c_2^T(ax_1 + bx_2) & = & ac_2^T x_1 + bc_2^T x_2 = a(0) + b(0) = 0 \\ \vdots & \vdots & \vdots \\ c_m^T(ax_1 + bx_2) & = & ac_m^T x_1 + bc_m^T x_2 = a(0) + b(0) = 0. \end{array}$$

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Theorem 2.2.3(2)

Theorem 2.2.3(2). Properties of Cross Product.
Let $x, y, z \in \mathbb{R}^3$ and $a \in \mathbb{R}$. Then:

$$x \times y = -y \times x \text{ (Anti-commutivity).}$$

Proof. Recall that $x \times y = [x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1]$.

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&= [x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2] \\
&= [-(x_2y_3 - x_3y_2), -(x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1)] \\
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