Theory of Matrices

Chapter 2. Vectors and Vector Spaces

2.3. Centered Vectors and Variances and Covariances of Vectors—Proofs of Theorems

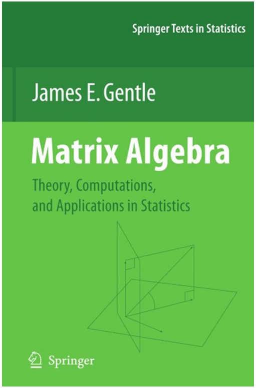

Table of contents

Theorem 2.3.1

Theorem 2.3.1. Properties of Covariance.
Let x, y, z be n-vectors and let $a \in \mathbb{R}$. Then:

1. $\text{Cov}(a1_n, y) = 0$,
2. $\text{Cov}(ax, y) = a\text{Cov}(x, y)$,
3. $\text{Cov}(y, y) = V(y)$.

Proof. We use the definition of covariance, $\text{Cov}(x, y) = \langle x - \overline{x}, y - \overline{y}\rangle/(n-1)$.
1. We have:

$$\begin{aligned} \text{Cov}(a1_n, y) &= \frac{\langle a1_n - a1_n, y - \overline{y}\rangle}{n-1} \text{ since } \overline{a1_n} = a1_n \\ &= \frac{\langle 0, y - \overline{y}\rangle}{n-1} = 0 \text{ by Theorem 2.1.6(1).} \end{aligned}$$

Theorem 2.3.1

Theorem 2.3.1. Properties of Covariance.
Let x, y, z be n-vectors and let $a \in \mathbb{R}$. Then:

1. $\text{Cov}(a1_n, y) = 0$,
2. $\text{Cov}(ax, y) = a\text{Cov}(x, y)$,
3. $\text{Cov}(y, y) = V(y)$.

Proof. We use the definition of covariance,
$\text{Cov}(x, y) = \langle x - \overline{x}, y - \overline{y} \rangle / (n - 1)$.
1. We have:

$$\begin{aligned} \text{Cov}(a1_n, y) &= \frac{\langle a1_n - a1_n, y - \overline{y} \rangle}{n-1} \text{ since } \overline{a1_n} = a1_n \\ &= \frac{\langle 0, y - \overline{y} \rangle}{n-1} = 0 \text{ by Theorem 2.1.6(1).} \end{aligned}$$

Theorem 2.3.1 (continued)

Proof (continued). We use the definition of covariance, $\text{Cov}(x,y) = \langle x - \overline{x}, y - \overline{y}\rangle/(n-1)$.

2. Notice that $\overline{ax} = \frac{\sum_{i=1}^n ax_i}{n} = a\frac{\sum_{i=1}^n x_i}{n} = a\overline{x}$, so

$$\begin{aligned}\text{Cov}(ax,y) &= \frac{\langle ax - \overline{ax}, y - \overline{y}\rangle}{n-1} = \frac{\langle ax - a\overline{x}, y - \overline{y}\rangle}{n-1} \\ &= \frac{\langle a(x - \overline{x}), y - \overline{y}\rangle}{n-1} = \frac{a\langle x - \overline{x}, y - \overline{y}\rangle}{n-1} \text{ by Thm 2.1.6(3)} \\ &= a\text{Cov}(x,y).\end{aligned}$$

Theorem 2.3.1 (continued)

Proof (continued). We use the definition of covariance, $\mathrm{Cov}(x,y) = \langle x - \overline{x}, y - \overline{y}\rangle/(n-1)$.

2. Notice that $\overline{ax} = \frac{\sum_{i=1}^n ax_i}{n} = a\frac{\sum_{i=1}^n x_i}{n} = a\overline{x}$, so

$$\begin{aligned}
\mathrm{Cov}(ax,y) &= \frac{\langle ax - \overline{ax}, y - \overline{y}\rangle}{n-1} = \frac{\langle ax - a\overline{x}, y - \overline{y}\rangle}{n-1} \\
&= \frac{\langle a(x - \overline{x}), y - \overline{y}\rangle}{n-1} = \frac{a\langle x - \overline{x}, y - \overline{y}\rangle}{n-1} \text{ by Thm 2.1.6(3)} \\
&= a\mathrm{Cov}(x,y).
\end{aligned}$$

3. The proof of this is in the class notes before the statement of the theorem. □

Theorem 2.3.1 (continued)

Proof (continued). We use the definition of covariance, $\mathrm{Cov}(x,y) = \langle x - \overline{x}, y - \overline{y}\rangle/(n-1)$.

2. Notice that $\overline{ax} = \frac{\sum_{i=1}^n ax_i}{n} = a\frac{\sum_{i=1}^n x_i}{n} = a\overline{x}$, so

$$\begin{aligned}
\mathrm{Cov}(ax,y) &= \frac{\langle ax - \overline{ax}, y - \overline{y}\rangle}{n-1} = \frac{\langle ax - a\overline{x}, y - \overline{y}\rangle}{n-1} \\
&= \frac{\langle a(x - \overline{x}), y - \overline{y}\rangle}{n-1} = \frac{a\langle x - \overline{x}, y - \overline{y}\rangle}{n-1} \text{ by Thm 2.1.6(3)} \\
&= a\mathrm{Cov}(x,y).
\end{aligned}$$

3. The proof of this is in the class notes before the statement of the theorem. □