Chapter 2. Vectors and Vector Spaces
2.3. Centered Vectors and Variances and Covariances of Vectors—Proofs of Theorems
1. Theorem 2.3.1. Properties of Covariance
Theorem 2.3.1

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Let \( x, y, z \) be \( n \)-vectors and let \( a \in \mathbb{R} \). Then:

1. \( \text{Cov}(a1_n, y) = 0 \),
2. \( \text{Cov}(ax, y) = a\text{Cov}(x, y) \),
3. \( \text{Cov}(y, y) = \text{V}(y) \).

**Proof.** We use the definition of covariance, 
\( \text{Cov}(x, y) = \langle x - \bar{x}, y - \bar{y} \rangle / (n - 1) \).

1. We have:

\[
\text{Cov}(a1_n, y) = \frac{\langle a1_n - a1_n, y - \bar{y} \rangle}{n - 1}
\]

since \( a1_n = a1_n \)

\[
= \frac{\langle 0, y - \bar{y} \rangle}{n - 1} = 0 \text{ by Theorem 2.1.6(1)}.
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Proof (continued). We use the definition of covariance, 
\[ \text{Cov}(x, y) = \frac{\langle x - \bar{x}, y - \bar{y} \rangle}{(n - 1)}. \]

2. Notice that 
\[ \bar{ax} = \frac{\sum_{i=1}^{n} ax_i}{n} = a \frac{\sum_{i=1}^{n} x_i}{n} = a\bar{x}, \]
so 
\[ \text{Cov}(ax, y) = \frac{\langle ax - \bar{a}x, y - \bar{y} \rangle}{n - 1} = \frac{\langle a(x - \bar{x}, y - \bar{y}) \rangle}{n - 1} = \frac{a\langle x - \bar{x}, y - \bar{y} \rangle}{n - 1} \]
by Theorem 2.1.6(3) 
\[ = a\text{Cov}(x, y). \]
Proof (continued). We use the definition of covariance, 
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\[ = \frac{\langle a(x - \bar{x}, y - \bar{y}) \rangle}{n - 1} = a \frac{\langle x - \bar{x}, y - \bar{y} \rangle}{n - 1} \text{ by Theorem 2.1.6(3)} \]
\[ = a \text{Cov}(x, y). \]

3. The proof of this is in the class notes before the statement of the theorem.
Theorem 2.3.1 (continued)

Proof (continued). We use the definition of covariance, $\text{Cov}(x, y) = \langle x - \overline{x}, y - \overline{y} \rangle/(n - 1)$.

2. Notice that $a\overline{x} = \frac{\sum_{i=1}^{n} ax_i}{n} = a \frac{\sum_{i=1}^{n} x_i}{n} = a\overline{x}$, so

$$\text{Cov}(ax, y) = \frac{\langle ax - a\overline{x}, y - \overline{y} \rangle}{n - 1} = a \frac{\langle x - \overline{x}, y - \overline{y} \rangle}{n - 1}$$

by Theorem 2.1.6(3)

$$= a\text{Cov}(x, y).$$

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