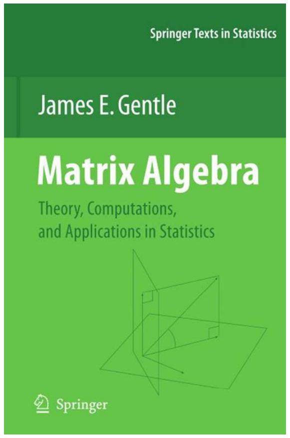
Theory of Matrices

Chapter 3. Basic Properties of Matrices

3.1. Basic Definitions and Notation—Proofs of Theorems



Theorem 3.1.1

Theorem 3.1.1. Suppose matrix A is diagonally dominant (that is, A is symmetric and row and column diagonally dominant). If B is a principal submatrix of A then B is also diagonally dominant.

Proof. Let $A = [a_{ij}]$ be symmetric and diagonally dominant. Let $B = [b_{k\ell}]$ be a principal submatrix of A. We need to show that B is symmetric and row diagonally dominant. Consider entry $b_{k\ell}$ in B. Then $b_{k\ell} = a_{ij}$ for some i, j. Now b_{kk} and $b_{\ell\ell}$ are on the diagonal of B and we have $b_{kk} = a_{ii}$ and $b_{\ell\ell} = a_{jj}$. So in producing submatrix B, neither row j nor column i of matrix A was eliminated and $a_{ji} = b_{\ell k}$. Since A is symmetric then $a_{ij} = a_{ji}$ and so $b_{k\ell} = b_{\ell k}$ and B is symmetric. For every b_{kk} in B we have $b_{kk} = a_{ii}$ for some a_{ii} in A, and since A is row diagonally dominant then $|b_{kk}| = |a_{ii}| > \sum_{j=1, j\neq i} |a_{ij}| \geq \sum_{\ell=1,\ell\neq k}^{m'} |b_{k\ell}|$ where m' is the number of columns in B. So B is row diagonally dominant, as claimed. □

Theorem 3.1.A

Theorem 3.1.A. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then $\det(A) = \det(A^T)$.

Proof. Let $A^T = [b_{ij}]$ so that $b_{ij} = a_{ji}$. For $\pi \in S_n$, consider $\prod_{i=1}^n a_{i\,\pi(i)}$. Since π is a permutation of $\{1, 2, \ldots, n\}$ then each index $1, 2, \ldots, n$ appears as the second index in the product (the index representing the column of the entry) so that $\prod_{i=1}^n a_{i\,\pi(i)} = \prod_{j=1}^n a_{\gamma(j)\,j}$ where γ is some element of S_n. Notice that if $i = \gamma(j)$ then $j = \pi(i)$. So in the group S_n, $\gamma = \pi^{-1}$. Now the even permutations in S_n form the subgroup A_n (the alternating group) and so the inverse of an even permutation is an even permutation. The $n!/2$ odd permutations in $S_n \setminus A_n$ must include all inverses in this set and so the inverse of an odd permutation is an odd permutation. Hence $\sigma(\gamma) = \sigma(\pi)$. Therefore $\sigma(\pi)\prod_{i=1}^n a_{i\,\pi(i)} = \sigma(\gamma)\prod_{j=1}^n a_{\gamma(j)\,j}$. In terms of b_{ij},

$$\sigma(\pi)\prod_{i=1}^n a_{i\,\pi(i)} = \sigma(\gamma)\prod_{j=1}^n b_{j\,\gamma(j)}. \qquad (*)$$

Theorem 3.1.A(continued)

Theorem 3.1.A. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then $\det(A) = \det(A^T)$.

Proof (continued). Summing over all permutations in S_n gives

$$\det(A) = \sum_{\pi\in S_n} \sigma(\pi)\prod_{i=1}^n a_{i\,\pi(i)} = \sum_{\gamma\in S_n} \sigma(\gamma)\prod_{j=1}^n b_{j\,\gamma(j)} = \det(A^T).$$

(Notice that the sums are the same since π and γ range over all elements of S_n. Equation $(*)$ does not claim $\pi = \gamma$ but instead, as we say, $\pi = \gamma^{-1}$.) □

Theorem 3.1.B

Theorem 3.1.B. If an $n \times n$ matrix B is formed from a $n \times n$ matrix A by multiplying all of the elements of one row or one column of A by the same scalar k (and leaving the elements of the other $n-1$ row or columns unchanged) then $\det(B) = k\det(A)$.

Proof. By definition, for $A = [a_{ij}]$ we have $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$. In the product $\prod_{i=1}^n a_{i\,\pi(i)}$ there is exactly one element from each row (since i ranges over $1, 2, \ldots, n$) and exactly one element from each column (since $\pi(i)$ ranges over $1, 2, \ldots, n$). So if B satisfies the hypotheses, then for given $\pi \in S_n$, we have $\prod_{i=1}^n b_{i\,\pi(i)} = k \prod_{i=1}^n a_{i\,\pi(i)}$ since exactly one $b_{i\,\pi(i)}$ equals $k a_{i\,\pi(i)}$ and for the other $n-1$ values of i, $b_{i\,\pi(i)} = a_{i\,\pi(i)}$. So $\det(B) =$

$$\sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n b_{i\,\pi(i)} = \sum_{\pi \in S_n} \sigma(\pi) k \prod_{i=1}^n a_{i\,\pi(i)} = k \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$$

$= k\det(A)$. □

Theorem 3.1.C

Theorem 3.1.C. If a $n \times n$ matrix $B = [b_{ij}]$ is formed from an $n \times n$ matrix $A = [a_{ij}]$ by interchanging two rows (or columns) of A then $\det(B) = -\det(A)$.

Proof. Suppose B is found by interchanging the ith and kth rows of A where $k > i$. We have $\det(B) = \prod_{\pi \in S_n} \sigma(\pi) \prod_{j=1}^n b_{j\,\pi(j)}$ where

$$\begin{aligned} \prod_{j=1}^n b_{j\,\pi(j)} &= b_{1\,\pi(1)} b_{2\,\pi(2)} \cdots b_{(i-1)\,\pi(i-1)} b_{i\,\pi(i)} b_{(i+1)\,\pi(i+1)} \cdots \\ &\quad b_{(k-1)\,\pi(k-1)} b_{k\,\pi(k)} b_{(k+1)\,\pi(k+1)} \cdots b_{n\,\pi(n)} \\ &= a_{1\,\pi(1)} a_{2\,\pi(2)} \cdots a_{(i-1)\,\pi(i-1)} a_{k\,\pi(i)} a_{(i+1)\,\pi(i+1)} \cdots \\ &\quad a_{(k-1)\,\pi(k-1)} a_{i\,\pi(k)} a_{(k+1)\,\pi(k+1)} \cdots a_{n\,\pi(n)} \end{aligned}$$

since $b_{i\,\pi(i)} = a_{k\,\pi(i)}$ and $b_{k\,\pi(k)} = a_{i\,\pi(k)}$.

Theorem 3.1.C (continued 1)

Proof. To swap indices i and k we define $\gamma \in S_n$ as
$\gamma(j) = \begin{cases} \pi(j) & \text{if } j \neq i, k \\ \pi(k) & \text{if } j = i \\ \pi(i) & \text{if } j = k. \end{cases}$ Then $\gamma = \pi \circ (i, k)$ and so γ can be written with one more transposition ("two cycle") than π; that is, the parity (even or odd) of γ is opposite of the parity of π. Therefore $\sigma(\pi) = -\sigma(\gamma)$. But as π ranges over S_n then $\gamma = \pi \circ (i, k)$ ranges over S_n (such γ's make up a row of the multiplication table ["Cayley table"] of S_n). So

$$\det(B) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{j=1}^n b_{j\,\pi(j)} = \sum_{\gamma \in S_n} -\sigma(\gamma) \prod_{j=1}^n a_{j\,\gamma(j)}$$

where $\gamma = \pi \circ (i, k)$. Hence

$$\det(B) = \sum_{\gamma \in S_n} -\sigma(\gamma) \prod_{j=1}^n a_{j\,\gamma(j)} = -\sum_{\gamma \in S_n} \sigma(\gamma) \prod_{j=1}^n a_{j\,\gamma(j)} = -\det(A).$$

Theorem 3.1.C (continued 2)

Theorem 3.1.C. If a $n \times n$ matrix $B = [b_{ij}]$ is formed from an $n \times n$ matrix $A = [a_{ij}]$ by interchanging two rows (or columns) of A then $\det(B) = -\det(A)$.

Proof. If B is formed by interchanging two columns of A then

$$\begin{aligned} \det(B) &= \det(B^T) \text{ by Theorem 3.1.A} \\ &= -\det(A^T) \text{ by above} \\ &= -\det(A) \text{ by Theorem 3.1.A.} \end{aligned}$$

□

Theorem 3.1.E

Theorem 3.1.E. Let B represent a matrix formed from $n \times n$ matrix A by adding to any row (or column) of A, scalar multiples of one or more other rows (or columns). Then $\det(B) = \det(A)$.

Proof. Let a_i and b_i be the ith rows of matrices A and B, respectively, where $a_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$ and $b_i = [b_{i1}, b_{i2}, \ldots, b_{in}]$ (remember, we don't notationally distinguish between representations of scalars and vectors). Then for some $s \in \mathbb{N}$, $1 \le s \le n$ and some scalars $k_1, k_2, \ldots, k_{s-1}, k_{s+1}, \ldots, k_n$ (possibly 0) we have that the sth row of B is $b_s = a_s + \sum_{j=1, j\neq s}^{n} k_j a_j$ and the ith row of B, where $i \neq s$, is $b_i = a_i$.

Theorem 3.1.E (continued)

Proof (continued). So

$$\begin{aligned}
\det(B) &= \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} b_{i\,\pi(i)} = \sum_{\pi \in S_n} \sigma(\pi) b_{s\,\pi(s)} \prod_{i=1, i\neq s} b_{i\,\pi(i)} \\
&= \sum_{\pi \in S_n} \sigma(\pi) \left(a_{s\,\pi(s)} + \sum_{j=1, j\neq s}^{n} k_j a_{j\,\pi(j)} \right) \prod_{i=1, i\neq s}^{n} a_{i\,\pi(i)} \\
&= \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)} + \sum_{j=1, j\neq s} \left(\sum_{\pi \in S_n} \sigma(\pi) k_j a_{j\,\pi(j)} \prod_{i=1, i\neq s} a_{i\,\pi(i)} \right) \\
&= \det(A) + \sum_{j=1, j\neq s} \det(B_j)
\end{aligned}$$

where B_j is the matrix formed from A by replacing the sth row of A with $k_j a_j$ (notice $j \neq s$). By Corollary 3.1.D, $\det(B_j) = 0$ for $j \neq s$ and so $\det(B) = \det(A)$ as claimed. By Theorem 3.1.A, the result also holds if we replace "row" with "column". □

Theorem 3.1.F

Theorem 3.1.F. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let α_{ij} represent the cofactor of a_{ij}. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}\alpha_{ij} \text{ for } i = 1, 2, \ldots, n, \qquad (5.1)$$

and

$$\det(A) = \sum_{i=1}^{n} a_{ij}\alpha_{ij} \text{ for } j = 1, 2, \ldots, . \qquad (5.2)$$

Proof. Let A_{ij} be the $(n-1) \times (n-1)$ matrix that is formed by eliminating the ith row and jth column of matrix A. Consider equation (5.1) for the case $i = 1$. Denote by $a_{ts}^{(j)}$ the (t, s)th element of A_{1j} (so t and s range over the set $\{1, 2, \ldots, n-1\}$). Then $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)}$. When $i = 1$ and π ranges over S_n, the value of $\pi(i)$ ranges over the set $\{1, 2, \ldots, n\}$.

Theorem 3.1.F (continued 1)

Proof (continued). Let $S_n^j \subset S_n$ denote all $\pi \in S_n$ such that $\pi(1) = j$ (so for given $j \in \{1, 2, \ldots, n\}$, the permutations in S_n^j all map 1 to j and map the remaining $n-1$ values $2, 3, \ldots, n$ to $1, 2, \ldots, j-1, j+1, j+2, \ldots, n$, so that $|S_n^j| = (n-1)!$ for each $j \in \{1, 2, \ldots, n\}$). We have $S_n = \cup_{j=1}^{n} S_n^j$ and so

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)} \right) \\
&= \sum_{j=1}^{n} \left(\sum_{\pi \in S_n^j} \sigma(\pi) a_{1j} \prod_{i=2}^{n} a_{i\,\pi(i)} \right) \text{ since } \pi(1) = j \\
&= \sum_{j=1}^{n} a_{1j} \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=2}^{n} a_{i\,\pi(i)} \right). \qquad (*)
\end{aligned}$$

Theorem 3.1.F (continued 2)

Proof (continued). Now in $(*)$, as permutation π ranges over S_n^j and as i ranges over $\{2,3,\ldots,n\}$, the elements $a_{i\,\pi(i)}$ ranges over all entries of A_{1j}. Since we denote the (t,s) entry of A_{1j} as $a_{ts}^{(j)}$, then we can re-index the product and inner summation in $(*)$ from $i \in \{2,3,\ldots,n\}$ and $\pi \in S_n^j$ to $t \in \{1,2,\ldots,n-1\}$ and $\gamma \in S_{n-1}$, respectively. We do so by defining $t = i-1$ for $i \in \{2,3,\ldots,n\}$ and $\gamma \in S_{n-1}$ as $\gamma(t) = \begin{cases} \pi(t+1) & \text{if } \pi(t+1) < j \\ \pi(t+1)-1 & \text{if } \pi(t+1) > j \end{cases}$ for $t \in \{1,2,\ldots,n-1\}$. We then have that $\gamma : \{1,2,\ldots,n-1\} \to \{1,2,\ldots,n-1\}$ and so $\gamma \in S_{n-1}$. Also, $\gamma(t) = \pi(i)$ if $\pi(i) < j$, and $\gamma(t) = \pi(i) - 1$ if $\pi(i) > j$. Now extend $\gamma \in S_{n-1}$ to $\gamma' \in S_n$, be defining $\gamma'(t) = \gamma(t)$ for $t \in \{1,2,\ldots,n-1\}$ and $\gamma'(n) = n$. Then $\sigma(\gamma') = \sigma(\gamma)$.

Theorem 3.1.F (continued 3)

Proof (continued). We can relate γ' and π with the following mapping:

$$\begin{array}{ccccc} \gamma'(1) & \gamma'(2) & \cdots & \gamma'(n-1) & \gamma'(n) \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \pi(2) & \pi(3) & \cdots & \pi(n) & \pi(1) = j. \end{array}$$

We will need to "move the jth term to the right end" and do so using the mapping π'':

$$\begin{array}{ccccccccc} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ 1 & 2 & \cdots & j-1 & n & j & \cdots & n-2 & n-1 \end{array}.$$

So first we increase indices by 1 (mod n) with the permutation

$$\pi' = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} = (2,3)(3,4)\cdots(n-1,n)(n,1),$$

second we apply permutation π, and third we perform the second mapping above using the permutation π'' where ...

Theorem 3.1.F (continued 4)

Proof (continued).

$$\pi'' = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\ 1 & 2 & \cdots & j-1 & n & j & \cdots & n-2 & n-1 \end{pmatrix}$$
$$= (n-2, n-1)(n-3, n-2)\cdots(j, j+1)(j, n).$$

Then $\gamma' = \pi''\pi\pi'$. Notice $\sigma(\pi') = (-1)^{n-1}$ and $\sigma(\pi'') = (-1)^{n-j}$, so that

$$\sigma(\gamma) = \sigma(\gamma') = \sigma(\pi''\pi\pi')$$
$$= \sigma(\pi'')\sigma(\pi)\sigma(\pi') = (-1)^{2n-j-1}\sigma(\pi) = (-1)^{j+1}\sigma(\pi),$$

or $\sigma(\pi) = (-1)^{j+1}\sigma(\gamma)$. So $(*)$ becomes

$$\det(A) = \sum_{j=1}^{n} a_{1j} \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=2}^{n} a_{i\,\pi(i)} \right) \text{ by } (*)$$

Theorem 3.1.F (continued 5)

Proof (continued).

$$\det(A) = \sum_{j=1}^{n} a_{1j} \left(\sum_{\gamma \in S_{n-1}} (-1)^{j+1}\sigma(\gamma) \prod_{t=1}^{n-1} a_{t\,\gamma(t)}^{(j)} \right)$$

where $\gamma' = \pi''\pi\pi'$ and γ is the restriction of γ' to $\{1,2,\ldots,n-1\}$

$$= \sum_{j=1}^{n} a_{1j}(-1)^{j+1} \left(\sum_{\gamma \in S_{n-1}} \sigma(\gamma) \prod_{t=1}^{n-1} a_{t\,\gamma(t)}^{(j)} \right)$$
$$= \sum_{j=1}^{n} a_{1j}(-1)^{j+1}\det(A_{1j}) = \sum_{j=1}^{n} a_{1j}\alpha_{1j},$$

and the claim holds for $i = 1$.

Theorem 3.1.F (continued 6)

Proof (continued). Consider now equation (5.1) for $i > 1$. Let B be the $n \times n$ matrix formed from A by interchanging the $(i-1)$th and ith rows, then the $(i-2)$th and $(i-1)$th rows, ..., then the 1st and 2nd rows (so that the first row of B is the ith row of A and the 2nd through ith row of B is the 1st through $(i-1)$th row of A, respectively). By Theorem 3.1.C, $\det(A) = (-1)^{i-1}\det(B)$. Let B_{1j} be the $(n-1)\times(n-1)$ matrix obtained by eliminating the 1st row and the jth column of B, and let b_{1j} be the jth element of the first row of B. Then $B_{1j} = A_{ij}$ and so

$$\begin{aligned}\det(A) &= (-1)^{i-1}\det(B) = (-1)^{i-1}\sum_{j=1}^{n} b_{1j}(-1)^{1+j}\det(B_{1j}) \\ &\quad \text{by the first part of the proof} \\ &= (-1)^{i-1}\sum_{j=1}^{n} a_{ij}(-1)^{1+j}\det(A_{ij}) \text{ since } B_{1j} = A_{ij}\end{aligned}$$

Theorem 3.1.F (continued 7)

Proof (continued).

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) = \sum_{j=1}^{n} a_{ij}\alpha_{ij}.$$

So equation (5.1) holds for all $i = 1, 2, \ldots, n$.

Finally, consider equation (5.2). Notice that the matrix formed by eliminating the jth row and ith column of A^T is A_{ij}^T. So the cofactor of the (j,i)th element of A^T is $(-1)^{j+i}\det(A_{ij}^T) = (-1)^{j+i}\det(A_{ij}) = \alpha_{ij}$ by Theorem 3.1.A. Since the (j,i)th element of A^T is the (i,j)th element of A, then by equation (5.1) and Theorem 3.1.A,

$$\begin{aligned}\det(A) &= \det(A^T) = \sum_{j=1}^{n} a'_{ij}\alpha'_{ij} \text{ where } a'_{ij} = a_{ji} \text{ and } \alpha'_{ij} = (-1)^{i+j}\det(A_{ij}^T) \\ &= \sum_{i=1}^{n} a'_{ji}\alpha'_{ji} \text{ interchanging } i \text{ and } j\end{aligned}$$

Theorem 3.1.F (continued 8)

Theorem 3.1.F. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let α_{ij} represent the cofactor of a_{ij}. Then

$$\det(A) = \sum_{i=1}^{n} a_{ij}\alpha_{ij} \text{ for } j = 1, 2, \ldots, . \qquad (5.2)$$

Proof (continued). ...

$$\begin{aligned}\det(A) &= \sum_{i=1}^{n} a_{ij}(-1)^{j+i}\det(A_{ji}^T) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) \\ &= \sum_{i=1}^{n} a_{ij}\alpha_{ij}\end{aligned}$$

and equation 5.2 holds. □

Theorem 3.1.3

Theorem 3.1.3. Let A be an $n \times n$ matrix with adjoint $\text{adj}(A) = [\alpha_{ij}]^T$. Then $A\,\text{adj}(A) = \text{adj}(A)A = \det(A)I_n$.

Proof. With $A = [a_{ij}]$ we have the (i,j) entry of $A\,\text{adj}(A)$ as $\sum_{k=1}^{n} a_{ik}\alpha_{jk}$. By Theorem 3.1.F, for $i = j$ this is $\det(A)$.

If $i \neq j$, consider the matrix $B = [b_{ij}]$ where B is $n \times n$ and has the same rows as A, except that its jth row is the same as the ith row of A. Then the cofactors α_{jk} of A are the same as the cofactors β_{jk} of B for $1 \le k \le n$. Also, since the jth row of B is the same as the ith of A then $b_{jk} = a_{ik}$ for $1 \le k \le n$. Since the ith row and the jth row are the same in B then, by Note 3.1.C, $\det(B) = 0$. So for $i \neq j$ the (i,j) entry of $A\,\text{adj}(A)$ is

$$\sum_{k=1}^{n} a_{ik}\alpha_{jk} = \sum_{k=1}^{n} b_{jk}\beta_{jk} = \det(B) = 0 \text{ by Theorem 3.1.F.}$$

So the (i,j) entry of $A\,\text{adj}(A)$ is $\det(A)$ for $i = j$ and 0 for $i \neq j$; that is $A\,\text{adj}(A) = \det(A)I_n$, as claimed. Similarly, $\text{adj}(A)A = \det(A)I_n$. □

Theorem 3.1.G

Theorem 3.1.G. Let T be an $m \times m$ matrix, V an $n \times m$ matrix, W an $n \times n$ matrix, and let '0' represent the $m \times n$ matrix of all entries as 0. Then the determinant of the partitioned matrix is

$$\det \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = \det \begin{bmatrix} W & V \\ 0 & T \end{bmatrix} = \det(T)\det(W).$$

Proof. Let $A = \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = [a_{ij}]$ be a partitioned $(m+n) \times (m+n)$ matrix. Let $T = [t_{ij}]$ and $W = [w_{ij}]$, so that $t_{ij} = a_{ij}$ for $i,j \in \{1,2,\ldots,m\}$ and $w_{ij} = a_{(i+m)\,(j+m)}$ for $i,j \in \{1,2,\ldots,n\}$. By definition: $\det(A) = \sum_{\pi \in S_{m+n}} \sigma(\pi) \prod_{i=1}^{m+n} a_{i\,\pi(i)}$. $\quad (*)$

Now the only time the product in $(*)$ *might* be nonzero is when π is a permutation mapping $\{1,2,\ldots,m\}$ to itself (otherwise $a_{i\,\pi(i)} = 0$ for some $i \in \{1,2,\ldots,m\}$), and hence also mapping $\{m+1,m+2,\ldots,m+n\}$ to itself. Denote all such permutations as S'_{m+n}.

Theorem 3.1.G (continued 1)

Proof (continued). Such $\pi \in S'_{m+n}$ can be written as the product of two permutations, π_m and π_n, in S'_{m+n} where π_m fixes $\{m+1,m+2,\ldots,m+n\}$ and π_n fixes $\{1,2,\ldots,m\}$; that is, $\pi = \pi_m\pi_n$ and $\sigma(\pi) = \sigma(\pi_m)\sigma(\pi_n)$. Now if we restrict π_m to $\{1,2,\ldots,m\}$ and denote the resulting function as π'_m then we have $\pi'_m \in S_m$. If we define $\pi'_n(i-m) = \pi_n(i) - m$ for $i \in \{m+1,m+2,\ldots,m+n\}$, then $\pi'_n : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ and $\pi'_n \in S_n$. We have $\sigma(\pi_m) = \sigma(\pi'_m)$ and $\sigma(\pi_n) = \sigma(\pi'_n)$. So from $(*)$ we have

$$\begin{aligned} \det(A) &= \sum_{\pi \in S'_{m+n}} \sigma(\pi) \prod_{i=1}^{m+n} a_{i\,\pi(i)} \\ &= \sum_{\pi_m,\pi_n \in S'_{m+n}} \sigma(\pi_m)\sigma(\pi_n) \prod_{i=1}^{m} a_{i\,\pi_m(i)} \prod_{i=m+1}^{m+n} a_{i\,\pi_n(i)} \end{aligned}$$

where each $\pi \in S'_{m+n}$ is written as $\pi = \pi_m\pi_n$

Theorem 3.1.G (continued 2)

Proof (continued).

$$\begin{aligned} \det(A) &= \sum_{\pi_m,\pi_n \in S'_{m+n}} \sigma(\pi_m)\sigma(\pi_n) \prod_{i=1}^{m} a_{i\,\pi_m(i)} \prod_{i=m+1}^{m+n} a_{i\,\pi_n(i)} \\ &= \sum_{\pi'_m \in S_m, \pi'_n \in S_n} \sigma(\pi'_m)\sigma(\pi'_n) \prod_{i=1}^{m} a_{i\,\pi'_m(i)} \prod_{i=1}^{n} a_{(i+m)\,\pi_n(i+m)} \\ &= \sum_{\pi'_m \in S_m, \pi'_n \in S_n} \sigma(\pi'_m)\sigma(\pi'_n) \prod_{i=1}^{m} a_{i\,\pi'_m(i)} \prod_{i=1}^{n} a_{(i+m)\,\pi'_n(i)+m} \end{aligned}$$

since $\pi'_n(i-m) = \pi_n(i) - m$ for $i \in \{m+1,m+2,\ldots,m+n\}$ or $\pi'_n(i) + m = \pi_n(i+m)$ for $i \in \{1,2,\ldots,n\}$

$$= \sum_{\pi'_m \in S_m, \pi'_n \in S_n} \sigma(\pi'_m)\sigma(\pi'_n) \prod_{i=1}^{m} t_{i\,\pi'_m(i)} \prod_{i=1}^{n} w_{i\,\pi'_n(i)}$$

Theorem 3.1.G (continued 3)

Proof (continued).

$$\begin{aligned} \det(A) &= \sum_{\pi'_m \in S_m, \pi'_n \in S_n} \sigma(\pi'_m)\sigma(\pi'_n) \prod_{i=1}^{m} t_{i\,\pi'_m(i)} \prod_{i=1}^{n} w_{i\,\pi'_n(i)} \\ &= \sum_{\pi'_m \in S_m} \sigma(\pi'_m) \prod_{i=1}^{m} t_{i\,\pi'_m(i)} \sum_{\pi'_n \in S_n} \sigma(\pi'_n) \prod_{i=1}^{n} w_{i\,\pi'_n(i)} \\ &= \det(T)\det(W). \end{aligned}$$

The proof that $\det \begin{bmatrix} W & V \\ 0 & T \end{bmatrix} = \det(T)\det(W)$ is similar. □

Theorem 3.1.H

Theorem 3.1.H. Let A be $n \times n$ and let T be an $n \times n$ upper or lower triangular matrix with entries of 1 along the diagonal. Then $\det(AT) = \det(TA) = \det(A)$.

Proof. Consider the case AT where T is lower triangular. Define T_i to be an $n \times n$ matrix formed from I_n by replacing the ith column of I_n with the ith column of T (for $1 \leq i \leq n$). Then $T = T_1 T_2 \cdots T_n$, as shown in Exercise 3.1.C, so $AT = AT_1 T_2 \cdots T_n$. Define $B_0 = A$ and $B_i = AT_1 T_2 \cdots T_i$ (for $1 \leq i \leq n$). Consider $B_{i-1} T_i$ for $1 \leq i \leq n$. Since all columns of T_i, except for the ith column, are the same as I_n then the columns of $B_{i-1} T_i$ are the same as the columns of B_{i-1}, except for the ith column. Let $t_{1i}, t_{2i}, \ldots, t_{ni}$ be the entries in the ith column of T_i (so $t_{1i} = t_{2i} = \cdots = t_{(i-1)i} = 0$ and $t_{ii} = 1$). Let $b_1, b_2, \ldots, b_n$ be the columns of B_{i-1}.

Theorem 3.1.H (continued)

Theorem 3.1.H. Let A be $n \times n$ and let T be an $n \times n$ upper or lower triangular matrix with entries of 1 along the diagonal. Then $\det(AT) = \det(TA) = \det(A)$.

Proof (continued). Then the entries of the ith column of $B_{i-1} T_i$ are

$$\sum_{k=1}^{n} b_{jk} t_{ki} = b_{ji} + \sum_{k=i+1}^{n} b_{jk} t_{ki} \text{ for } 1 \leq j \leq n$$

where the entries of b_i are $b_{1i}, b_{2i}, \ldots, b_{ni}$. So the ith column of $B_{i-1} T_i$ is $b_i + \sum_{k=i+1}^{n} b_k t_{ki}$, which is the ith column of B_{i-1} plus a series of scalar multiples of the columns $b_{i+1}, b_{i+2}, \ldots, b_n$ of B_{i-1}. So by Theorem 3.1.E, $\det(B_i) = \det(B_{i-1} T_i) = \det(B_{i-1})$. This holds for $1 \leq i \leq n$, so

$$\det(A) = \det(B_0) = \det(B_1) = \det(B_2) = \cdots = \det(B_n) = \det(AT).$$

The result holds similarly for T upper triangular and for TA. □