Theory of Matrices

Chapter 3. Basic Properties of Matrices

3.1. Basic Definitions and Notation—Proofs of Theorems

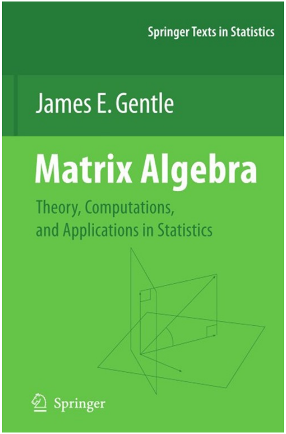

Table of contents

1. Theorem 3.1.1
2. Theorem 3.1.A
3. Theorem 3.1.B
4. Theorem 3.1.C
5. Theorem 3.1.E
6. Theorem 3.1.F
7. Theorem 3.1.3
8. Theorem 3.1.G
9. Theorem 3.1.H

Theorem 3.1.1

Theorem 3.1.1. Suppose matrix A is diagonally dominant (that is, A is symmetric and row and column diagonally dominant). If B is a principal submatrix of A then B is also diagonally dominant.

Proof. Let $A = [a_{ij}]$ be symmetric and diagonally dominant. Let $B = [b_{k\ell}]$ be a principal submatrix of A. We need to show that B is symmetric and row diagonally dominant.

Theorem 3.1.1

Theorem 3.1.1. Suppose matrix A is diagonally dominant (that is, A is symmetric and row and column diagonally dominant). If B is a principal submatrix of A then B is also diagonally dominant.

Proof. Let $A = [a_{ij}]$ be symmetric and diagonally dominant. Let $B = [b_{k\ell}]$ be a principal submatrix of A. We need to show that B is symmetric and row diagonally dominant. Consider entry $b_{k\ell}$ in B. Then $b_{k\ell} = a_{ij}$ for some i, j. Now b_{kk} and $b_{\ell\ell}$ are on the diagonal of B and we have $b_{kk} = a_{ii}$ and $b_{\ell\ell} = a_{jj}$. So in producing submatrix B, neither row j nor column i of matrix A was eliminated and $a_{ji} = b_{\ell k}$.

Theorem 3.1.1

Theorem 3.1.1. Suppose matrix A is diagonally dominant (that is, A is symmetric and row and column diagonally dominant). If B is a principal submatrix of A then B is also diagonally dominant.

Proof. Let $A = [a_{ij}]$ be symmetric and diagonally dominant. Let $B = [b_{k\ell}]$ be a principal submatrix of A. We need to show that B is symmetric and row diagonally dominant. Consider entry $b_{k\ell}$ in B. Then $b_{k\ell} = a_{ij}$ for some i, j. Now b_{kk} and $b_{\ell\ell}$ are on the diagonal of B and we have $b_{kk} = a_{ii}$ and $b_{\ell\ell} = a_{jj}$. So in producing submatrix B, neither row j nor column i of matrix A was eliminated and $a_{ji} = b_{\ell k}$. Since A is symmetric then $a_{ij} = a_{ji}$ and so $b_{k\ell} = b_{\ell k}$ and B is symmetric. For every b_{kk} in B we have $b_{kk} = a_{ii}$ for some a_{ii} in A, and since A is row diagonally dominant then

$$|b_{kk}| = |a_{ii}| > \sum_{j=1, j\neq i} |a_{ij}| \geq \sum_{\ell=1, \ell\neq k}^{m'} |b_{k\ell}|$$ where m' is the number of columns in B. So B is row diagonally dominant, as claimed. □

Theorem 3.1.1

Theorem 3.1.1. Suppose matrix A is diagonally dominant (that is, A is symmetric and row and column diagonally dominant). If B is a principal submatrix of A then B is also diagonally dominant.

Proof. Let $A = [a_{ij}]$ be symmetric and diagonally dominant. Let $B = [b_{k\ell}]$ be a principal submatrix of A. We need to show that B is symmetric and row diagonally dominant. Consider entry $b_{k\ell}$ in B. Then $b_{k\ell} = a_{ij}$ for some i, j. Now b_{kk} and $b_{\ell\ell}$ are on the diagonal of B and we have $b_{kk} = a_{ii}$ and $b_{\ell\ell} = a_{jj}$. So in producing submatrix B, neither row j nor column i of matrix A was eliminated and $a_{ji} = b_{\ell k}$. Since A is symmetric then $a_{ij} = a_{ji}$ and so $b_{k\ell} = b_{\ell k}$ and B is symmetric. For every b_{kk} in B we have $b_{kk} = a_{ii}$ for some a_{ii} in A, and since A is row diagonally dominant then

$|b_{kk}| = |a_{ii}| > \sum_{j=1, j\neq i} |a_{ij}| \geq \sum_{\ell=1,\ell\neq k}^{m'} |b_{k\ell}|$ where m' is the number of columns in B. So B is row diagonally dominant, as claimed. □

Theorem 3.1.A

Theorem 3.1.A. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then $\det(A) = \det(A^T)$.

Proof. Let $A^T = [b_{ij}]$ so that $b_{ij} = a_{ji}$. For $\pi \in S_n$, consider $\prod_{i=1}^{n} a_{i\,\pi(i)}$. Since π is a permutation of $\{1, 2, \ldots, n\}$ then each index $1, 2, \ldots, n$ appears as the second index in the product (the index representing the column of the entry) so that $\prod_{i=1}^{n} a_{i\,\pi(i)} = \prod_{j=1}^{n} a_{\gamma(j)\,j}$ where γ is some element of S_n.

Theorem 3.1.A

Theorem 3.1.A. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then $\det(A) = \det(A^T)$.

Proof. Let $A^T = [b_{ij}]$ so that $b_{ij} = a_{ji}$. For $\pi \in S_n$, consider $\prod_{i=1}^n a_{i\,\pi(i)}$. Since π is a permutation of $\{1, 2, \ldots, n\}$ then each index $1, 2, \ldots, n$ appears as the second index in the product (the index representing the column of the entry) so that $\prod_{i=1}^n a_{i\,\pi(i)} = \prod_{j=1}^n a_{\gamma(j)\,j}$ where γ is some element of S_n. Notice that if $i = \gamma(j)$ then $j = \pi(i)$. So in the group S_n, $\gamma = \pi^{-1}$. Now the even permutations in S_n form the subgroup A_n (the alternating group) and so the inverse of an even permutation is an even permutation. The $n!/2$ odd permutations in $S_n \setminus A_n$ must include all inverses in this set and so the inverse of an odd permutation is an odd permutation.

Theorem 3.1.A

Theorem 3.1.A. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then $\det(A) = \det(A^T)$.

Proof. Let $A^T = [b_{ij}]$ so that $b_{ij} = a_{ji}$. For $\pi \in S_n$, consider $\prod_{i=1}^n a_{i\,\pi(i)}$. Since π is a permutation of $\{1, 2, \ldots, n\}$ then each index $1, 2, \ldots, n$ appears as the second index in the product (the index representing the column of the entry) so that $\prod_{i=1}^n a_{i\,\pi(i)} = \prod_{j=1}^n a_{\gamma(j)\,j}$ where γ is some element of S_n. Notice that if $i = \gamma(j)$ then $j = \pi(i)$. So in the group S_n, $\gamma = \pi^{-1}$. Now the even permutations in S_n form the subgroup A_n (the alternating group) and so the inverse of an even permutation is an even permutation. The $n!/2$ odd permutations in $S_n \setminus A_n$ must include all inverses in this set and so the inverse of an odd permutation is an odd permutation. Hence $\sigma(\gamma) = \sigma(\pi)$. Therefore $\sigma(\pi)\prod_{i=1}^n a_{i\,\pi(i)} = \sigma(\gamma)\prod_{j=1}^n a_{\gamma(j)\,j}$. In terms of b_{ij},

$$\sigma(\pi)\prod_{i=1}^{n} a_{i\,\pi(i)} = \sigma(\gamma)\prod_{j=1}^{n} b_{j\,\gamma(j)}. \qquad (*)$$

Theorem 3.1.A

Theorem 3.1.A. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then $\det(A) = \det(A^T)$.

Proof. Let $A^T = [b_{ij}]$ so that $b_{ij} = a_{ji}$. For $\pi \in S_n$, consider $\prod_{i=1}^n a_{i\,\pi(i)}$. Since π is a permutation of $\{1, 2, \ldots, n\}$ then each index $1, 2, \ldots, n$ appears as the second index in the product (the index representing the column of the entry) so that $\prod_{i=1}^n a_{i\,\pi(i)} = \prod_{j=1}^n a_{\gamma(j)\,j}$ where γ is some element of S_n. Notice that if $i = \gamma(j)$ then $j = \pi(i)$. So in the group S_n, $\gamma = \pi^{-1}$. Now the even permutations in S_n form the subgroup A_n (the alternating group) and so the inverse of an even permutation is an even permutation. The $n!/2$ odd permutations in $S_n \setminus A_n$ must include all inverses in this set and so the inverse of an odd permutation is an odd permutation. Hence $\sigma(\gamma) = \sigma(\pi)$. Therefore $\sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)} = \sigma(\gamma) \prod_{j=1}^n a_{\gamma(j)\,j}$. In terms of b_{ij},

$$\sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)} = \sigma(\gamma) \prod_{j=1}^{n} b_{j\,\gamma(j)}. \qquad (*)$$

Theorem 3.1.A(continued)

Theorem 3.1.A. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then $\det(A) = \det(A^T)$.

Proof (continued). Summing over all permutations in S_n gives

$$\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)} = \sum_{\gamma \in S_n} \sigma(\gamma) \prod_{j=1}^{n} b_{j\,\gamma(j)} = \det(A^T).$$

(Notice that the sums are the same since π and γ range over all elements of S_n. Equation $(*)$ does not claim $\pi = \gamma$ but instead, as we say, $\pi = \gamma^{-1}$.) □

Theorem 3.1.B

Theorem 3.1.B. If an $n \times n$ matrix B is formed from a $n \times n$ matrix A by multiplying all of the elements of one row or one column of A by the same scalar k (and leaving the elements of the other $n-1$ row or columns unchanged) then $\det(B) = k \det(A)$.

Proof. By definition, for $A = [a_{ij}]$ we have $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$. In the product $\prod_{i=1}^n a_{i\,\pi(i)}$ there is exactly one element from each row (since i ranges over $1, 2, \ldots, n$) and exactly one element from each column (since $\pi(i)$ ranges over $1, 2, \ldots, n$).

Theorem 3.1.B

Theorem 3.1.B. If an $n \times n$ matrix B is formed from a $n \times n$ matrix A by multiplying all of the elements of one row or one column of A by the same scalar k (and leaving the elements of the other $n-1$ row or columns unchanged) then $\det(B) = k \det(A)$.

Proof. By definition, for $A = [a_{ij}]$ we have $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$. In the product $\prod_{i=1}^n a_{i\,\pi(i)}$ there is exactly one element from each row (since i ranges over $1, 2, \ldots, n$) and exactly one element from each column (since $\pi(i)$ ranges over $1, 2, \ldots, n$). So if B satisfies the hypotheses, then for given $\pi \in S_n$, we have $\prod_{i=1}^n b_{i\,\pi(i)} = k \prod_{i=1}^n a_{i\,\pi(i)}$ since exactly one $b_{i\,\pi(i)}$ equals $ka_{i\,\pi(i)}$ and for the other $n-1$ values of i, $b_{i\,\pi(i)} = a_{i\,\pi(i)}$.

Theorem 3.1.B

Theorem 3.1.B. If an $n \times n$ matrix B is formed from a $n \times n$ matrix A by multiplying all of the elements of one row or one column of A by the same scalar k (and leaving the elements of the other $n-1$ row or columns unchanged) then $\det(B) = k\det(A)$.

Proof. By definition, for $A = [a_{ij}]$ we have $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$. In the product $\prod_{i=1}^n a_{i\,\pi(i)}$ there is exactly one element from each row (since i ranges over $1, 2, \ldots, n$) and exactly one element from each column (since $\pi(i)$ ranges over $1, 2, \ldots, n$). So if B satisfies the hypotheses, then for given $\pi \in S_n$, we have $\prod_{i=1}^n b_{i\,\pi(i)} = k \prod_{i=1}^n a_{i\,\pi(i)}$ since exactly one $b_{i\,\pi(i)}$ equals $ka_{i\,\pi(i)}$ and for the other $n-1$ values of i, $b_{i\,\pi(i)} = a_{i\,\pi(i)}$. So $\det(B) =$

$$\sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n b_{i\,\pi(i)} = \sum_{\pi \in S_n} \sigma(\pi) k \prod_{i=1}^n a_{i\,\pi(i)} = k \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$$

$= k\det(A)$. □

Theorem 3.1.B

Theorem 3.1.B. If an $n \times n$ matrix B is formed from a $n \times n$ matrix A by multiplying all of the elements of one row or one column of A by the same scalar k (and leaving the elements of the other $n-1$ row or columns unchanged) then $\det(B) = k\det(A)$.

Proof. By definition, for $A = [a_{ij}]$ we have $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$. In the product $\prod_{i=1}^n a_{i\,\pi(i)}$ there is exactly one element from each row (since i ranges over $1, 2, \ldots, n$) and exactly one element from each column (since $\pi(i)$ ranges over $1, 2, \ldots, n$). So if B satisfies the hypotheses, then for given $\pi \in S_n$, we have $\prod_{i=1}^n b_{i\,\pi(i)} = k \prod_{i=1}^n a_{i\,\pi(i)}$ since exactly one $b_{i\,\pi(i)}$ equals $ka_{i\,\pi(i)}$ and for the other $n-1$ values of i, $b_{i\,\pi(i)} = a_{i\,\pi(i)}$. So $\det(B) =$

$$\sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n b_{i\,\pi(i)} = \sum_{\pi \in S_n} \sigma(\pi) k \prod_{i=1}^n a_{i\,\pi(i)} = k \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)}$$

$= k\det(A)$. □

Theorem 3.1.C

Theorem 3.1.C. If a $n \times n$ matrix $B = [b_{ij}]$ is formed from an $n \times n$ matrix $A = [a_{ij}]$ by interchanging two rows (or columns) of A then $\det(B) = -\det(A)$.

Proof. Suppose B is found by interchanging the ith and kth rows of A where $k > i$.

Theorem 3.1.C

Theorem 3.1.C. If a $n \times n$ matrix $B = [b_{ij}]$ is formed from an $n \times n$ matrix $A = [a_{ij}]$ by interchanging two rows (or columns) of A then $\det(B) = -\det(A)$.

Proof. Suppose B is found by interchanging the ith and kth rows of A where $k > i$. We have $\det(B) = \prod_{\pi \in S_n} \sigma(\pi) \prod_{j=1}^n b_{j\,\pi(j)}$ where

$$\begin{aligned}
\prod_{j=1}^n b_{j\,\pi(j)} &= b_{1\,\pi(1)} b_{2\,\pi(2)} \cdots b_{(i-1)\,\pi(i-1)} b_{i\,\pi(i)} b_{(i+1)\,\pi(i+1)} \cdots \\
& \quad b_{(k-1)\,\pi(k-1)} b_{k\,\pi(k)} b_{(k+1)\,\pi(k+1)} \cdots b_{n\,\pi(n)} \\
&= a_{1\,\pi(1)} a_{2\,\pi(2)} \cdots a_{(i-1)\,\pi(i-1)} a_{k\,\pi(i)} a_{(i+1)\,\pi(i+1)} \cdots \\
& \quad a_{(k-1)\,\pi(k-1)} a_{i\,\pi(k)} a_{(k+1)\,\pi(k+1)} \cdots a_{n\,\pi(n)} \\
& \quad \text{since } b_{i\,\pi(i)} = a_{k\,\pi(i)} \text{ and } b_{k\,\pi(k)} = a_{i\,\pi(k)}.
\end{aligned}$$

Theorem 3.1.C

Theorem 3.1.C. If a $n \times n$ matrix $B = [b_{ij}]$ is formed from an $n \times n$ matrix $A = [a_{ij}]$ by interchanging two rows (or columns) of A then $\det(B) = -\det(A)$.

Proof. Suppose B is found by interchanging the ith and kth rows of A where $k > i$. We have $\det(B) = \prod_{\pi \in S_n} \sigma(\pi) \prod_{j=1}^n b_{j\,\pi(j)}$ where

$$\begin{aligned}
\prod_{j=1}^{n} b_{j\,\pi(j)} &= b_{1\,\pi(1)} b_{2\,\pi(2)} \cdots b_{(i-1)\,\pi(i-1)} b_{i\,\pi(i)} b_{(i+1)\,\pi(i+1)} \cdots \\
& \quad b_{(k-1)\,\pi(k-1)} b_{k\,\pi(k)} b_{(k+1)\,\pi(k+1)} \cdots b_{n\,\pi(n)} \\
&= a_{1\,\pi(1)} a_{2\,\pi(2)} \cdots a_{(i-1)\,\pi(i-1)} a_{k\,\pi(i)} a_{(i+1)\,\pi(i+1)} \cdots \\
& \quad a_{(k-1)\,\pi(k-1)} a_{i\,\pi(k)} a_{(k+1)\,\pi(k+1)} \cdots a_{n\,\pi(n)} \\
& \quad \text{since } b_{i\,\pi(i)} = a_{k\,\pi(i)} \text{ and } b_{k\,\pi(k)} = a_{i\,\pi(k)}.
\end{aligned}$$

Theorem 3.1.C (continued 1)

Proof. To swap indices i and k we define $\gamma \in S_n$ as
$$\gamma(j) = \begin{cases} \pi(j) & \text{if } j \neq i, k \\ \pi(k) & \text{if } j = i \\ \pi(i) & \text{if } j = k. \end{cases}$$
Then $\gamma = \pi \circ (i, k)$ and so γ can be written with one more transposition ("two cycle") than π; that is, the parity (even or odd) of γ is opposite of the parity of π. Therefore $\sigma(\pi) = -\sigma(\gamma)$. But as π ranges over S_n then $\gamma = \pi \circ (i, k)$ ranges over S_n (such γ's make up a row of the multiplication table ["Cayley table"] of S_n). So

$$\det(B) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{j=1}^{n} b_{j\,\pi(j)} = \sum_{\gamma \in S_n} -\sigma(\gamma) \prod_{j=1}^{n} a_{j\,\gamma(j)}$$

where $\gamma = \pi \circ (i, k)$.

Theorem 3.1.C (continued 1)

Proof. To swap indices i and k we define $\gamma \in S_n$ as $\gamma(j) = \begin{cases} \pi(j) & \text{if } j \neq i, k \\ \pi(k) & \text{if } j = i \\ \pi(i) & \text{if } j = k. \end{cases}$ Then $\gamma = \pi \circ (i, k)$ and so γ can be written with one more transposition ("two cycle") than π; that is, the parity (even or odd) of γ is opposite of the parity of π. Therefore $\sigma(\pi) = -\sigma(\gamma)$. But as π ranges over S_n then $\gamma = \pi \circ (i, k)$ ranges over S_n (such γ's make up a row of the multiplication table ["Cayley table"] of S_n). So

$$\det(B) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{j=1}^{n} b_{j\,\pi(j)} = \sum_{\gamma \in S_n} -\sigma(\gamma) \prod_{j=1}^{n} a_{j\,\gamma(j)}$$

where $\gamma = \pi \circ (i, k)$. Hence

$$\det(B) = \sum_{\gamma \in S_n} -\sigma(\gamma) \prod_{j=1}^{n} a_{j\,\gamma(j)} = -\sum_{\gamma \in S_n} \sigma(\gamma) \prod_{j=1}^{n} a_{j\,\gamma(j)} = -\det(A).$$

Theorem 3.1.C (continued 1)

Proof. To swap indices i and k we define $\gamma \in S_n$ as

$$\gamma(j) = \begin{cases} \pi(j) & \text{if } j \neq i, k \\ \pi(k) & \text{if } j = i \\ \pi(i) & \text{if } j = k. \end{cases}$$

Then $\gamma = \pi \circ (i, k)$ and so γ can be written with one more transposition ("two cycle") than π; that is, the parity (even or odd) of γ is opposite of the parity of π. Therefore $\sigma(\pi) = -\sigma(\gamma)$. But as π ranges over S_n then $\gamma = \pi \circ (i, k)$ ranges over S_n (such γ's make up a row of the multiplication table ["Cayley table"] of S_n). So

$$\det(B) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{j=1}^{n} b_{j\,\pi(j)} = \sum_{\gamma \in S_n} -\sigma(\gamma) \prod_{j=1}^{n} a_{j\,\gamma(j)}$$

where $\gamma = \pi \circ (i, k)$. Hence

$$\det(B) = \sum_{\gamma \in S_n} -\sigma(\gamma) \prod_{j=1}^{n} a_{j\,\gamma(j)} = -\sum_{\gamma \in S_n} \sigma(\gamma) \prod_{j=1}^{n} a_{j\,\gamma(j)} = -\det(A).$$

Theorem 3.1.C (continued 2)

Theorem 3.1.C. If a $n \times n$ matrix $B = [b_{ij}]$ is formed from an $n \times n$ matrix $A = [a_{ij}]$ by interchanging two rows (or columns) of A then $\det(B) = -\det(A)$.

Proof. If B is formed by interchanging two columns of A then

$$\begin{aligned} \det(B) &= \det(B^T) \text{ by Theorem 3.1.A} \\ &= -\det(A^T) \text{ by above} \\ &= -\det(A) \text{ by Theorem 3.1.A.} \end{aligned}$$

□

Theorem 3.1.E

Theorem 3.1.E. Let B represent a matrix formed from $n \times n$ matrix A by adding to any row (or column) of A, scalar multiples of one or more other rows (or columns). Then $\det(B) = \det(A)$.

Proof. Let a_i and b_i be the ith rows of matrices A and B, respectively, where $a_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$ and $b_i = [b_{i1}, b_{i2}, \ldots, b_{in}]$ (remember, we don't notationally distinguish between representations of scalars and vectors). Then for some $s \in \mathbb{N}$, $1 \leq s \leq n$ and some scalars $k_1, k_2, \ldots, k_{s-1}, k_{s+1}, \ldots, k_n$ (possibly 0) we have that the sth row of B is $b_s = a_s + \sum_{j=1, j\neq s}^{n} k_j a_j$ and the ith row of B, where $i \neq s$, is $b_i = a_i$.

Theorem 3.1.E

Theorem 3.1.E. Let B represent a matrix formed from $n \times n$ matrix A by adding to any row (or column) of A, scalar multiples of one or more other rows (or columns). Then $\det(B) = \det(A)$.

Proof. Let a_i and b_i be the ith rows of matrices A and B, respectively, where $a_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$ and $b_i = [b_{i1}, b_{i2}, \ldots, b_{in}]$ (remember, we don't notationally distinguish between representations of scalars and vectors). Then for some $s \in \mathbb{N}$, $1 \le s \le n$ and some scalars $k_1, k_2, \ldots, k_{s-1}, k_{s+1}, \ldots, k_n$ (possibly 0) we have that the sth row of B is $b_s = a_s + \sum_{j=1, j \ne s}^{n} k_j a_j$ and the ith row of B, where $i \ne s$, is $b_i = a_i$.

Theorem 3.1.E (continued)

Proof (continued). So

$$\begin{aligned}
\det(B) &= \sum_{\pi\in S_n}\sigma(\pi)\prod_{i=1}^{n} b_{i\,\pi(i)} = \sum_{\pi\in S_n}\sigma(\pi) b_{s\,\pi(s)} \prod_{i=1, i\neq s} b_{i\,\pi(i)} \\
&= \sum_{\pi\in S_n}\sigma(\pi)\left(a_{s\,\pi(s)} + \sum_{j=1, j\neq s}^{n} k_j a_{j\,\pi(j)}\right)\prod_{i=1, i\neq s}^{n} a_{i\,\pi(i)} \\
&= \sum_{\pi\in S_n}\sigma(\pi)\prod_{i=1}^{n} a_{i\,\pi(i)} + \sum_{j=1, j\neq s}\left(\sum_{\pi\in S_n}\sigma(\pi) k_j a_{j\,\pi(j)} \prod_{i=1, i\neq s} a_{i\,\pi(i)}\right) \\
&= \det(A) + \sum_{j=1, j\neq s} \det(B_j)
\end{aligned}$$

where B_j is the matrix formed from A by replacing the sth row of A with $k_j a_j$ (notice $j \neq s$). By Corollary 3.1.D, $\det(B_j) = 0$ for $j \neq s$ and so $\det(B) = \det(A)$ as claimed. By Theorem 3.1.A, the result also holds if we replace "row" with "column". □

Theorem 3.1.E (continued)

Proof (continued). So

$$\begin{aligned}
\det(B) &= \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} b_{i\,\pi(i)} = \sum_{\pi \in S_n} \sigma(\pi) b_{s\,\pi(s)} \prod_{i=1, i\neq s} b_{i\,\pi(i)} \\
&= \sum_{\pi \in S_n} \sigma(\pi) \left(a_{s\,\pi(s)} + \sum_{j=1, j\neq s}^{n} k_j a_{j\,\pi(j)} \right) \prod_{i=1, i\neq s}^{n} a_{i\,\pi(i)} \\
&= \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)} + \sum_{j=1, j\neq s} \left(\sum_{\pi \in S_n} \sigma(\pi) k_j a_{j\,\pi(j)} \prod_{i=1, i\neq s} a_{i\,\pi(i)} \right) \\
&= \det(A) + \sum_{j=1, j\neq s} \det(B_j)
\end{aligned}$$

where B_j is the matrix formed from A by replacing the sth row of A with $k_j a_j$ (notice $j \neq s$). By Corollary 3.1.D, $\det(B_j) = 0$ for $j \neq s$ and so $\det(B) = \det(A)$ as claimed. By Theorem 3.1.A, the result also holds if we replace "row" with "column". □

Theorem 3.1.F

Theorem 3.1.F. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let α_{ij} represent the cofactor of a_{ij}. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}\alpha_{ij} \text{ for } i = 1, 2, \ldots, n, \qquad (5.1)$$

and

$$\det(A) = \sum_{i=1}^{n} a_{ij}\alpha_{ij} \text{ for } j = 1, 2, \ldots, . \qquad (5.2)$$

Proof. Let A_{ij} be the $(n-1) \times (n-1)$ matrix that is formed by eliminating the ith row and jth column of matrix A. Consider equation (5.1) for the case $i = 1$. Denote by $a_{ts}^{(j)}$ the (t, s)th element of A_{1j} (so t and s range over the set $\{1, 2, \ldots, n-1\}$).

Theorem 3.1.F

Theorem 3.1.F. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let α_{ij} represent the cofactor of a_{ij}. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}\alpha_{ij} \text{ for } i = 1, 2, \ldots, n, \qquad (5.1)$$

and

$$\det(A) = \sum_{i=1}^{n} a_{ij}\alpha_{ij} \text{ for } j = 1, 2, \ldots, . \qquad (5.2)$$

Proof. Let A_{ij} be the $(n-1) \times (n-1)$ matrix that is formed by eliminating the ith row and jth column of matrix A. Consider equation (5.1) for the case $i = 1$. Denote by $a_{ts}^{(j)}$ the (t, s)th element of A_{1j} (so t and s range over the set $\{1, 2, \ldots, n-1\}$). Then $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)}$. When $i = 1$ and π ranges over S_n, the value of $\pi(i)$ ranges over the set $\{1, 2, \ldots, n\}$.

Theorem 3.1.F

Theorem 3.1.F. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let α_{ij} represent the cofactor of a_{ij}. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}\alpha_{ij} \text{ for } i = 1, 2, \ldots, n, \qquad (5.1)$$

and

$$\det(A) = \sum_{i=1}^{n} a_{ij}\alpha_{ij} \text{ for } j = 1, 2, \ldots, . \qquad (5.2)$$

Proof. Let A_{ij} be the $(n-1) \times (n-1)$ matrix that is formed by eliminating the ith row and jth column of matrix A. Consider equation (5.1) for the case $i = 1$. Denote by $a_{ts}^{(j)}$ the (t, s)th element of A_{1j} (so t and s range over the set $\{1, 2, \ldots, n-1\}$). Then $\det(A) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)}$. When $i = 1$ and π ranges over S_n, the value of $\pi(i)$ ranges over the set $\{1, 2, \ldots, n\}$.

Theorem 3.1.F (continued 1)

Proof (continued). Let $S_n^j \subset S_n$ denote all $\pi \in S_n$ such that $\pi(1) = j$ (so for given $j \in \{1, 2, \ldots, n\}$, the permutations in S_n^j all map 1 to j and map the remaining $n-1$ values $2, 3, \ldots, n$ to $1, 2, \ldots, j-1, j+1, j+2, \ldots, n$, so that $|S_n^j| = (n-1)!$ for each $j \in \{1, 2, \ldots, n\}$). We have $S_n = \cup_{j=1}^n S_n^j$ and so

$$\begin{aligned}
\det(A) &= \sum_{j=1}^n \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=1}^n a_{i\,\pi(i)} \right) \\
&= \sum_{j=1}^n \left(\sum_{\pi \in S_n^j} \sigma(\pi) a_{1j} \prod_{i=2}^n a_{i\,\pi(i)} \right) \text{ since } \pi(1) = j \\
&= \sum_{j=1}^n a_{1j} \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=2}^n a_{i\,\pi(i)} \right). \quad (*)
\end{aligned}$$

Theorem 3.1.F (continued 1)

Proof (continued). Let $S_n^j \subset S_n$ denote all $\pi \in S_n$ such that $\pi(1) = j$ (so for given $j \in \{1, 2, \ldots, n\}$, the permutations in S_n^j all map 1 to j and map the remaining $n-1$ values $2, 3, \ldots, n$ to $1, 2, \ldots, j-1, j+1, j+2, \ldots, n$, so that $|S_n^j| = (n-1)!$ for each $j \in \{1, 2, \ldots, n\}$). We have $S_n = \dot\cup_{j=1}^n S_n^j$ and so

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=1}^{n} a_{i\,\pi(i)} \right) \\
&= \sum_{j=1}^{n} \left(\sum_{\pi \in S_n^j} \sigma(\pi) a_{1j} \prod_{i=2}^{n} a_{i\,\pi(i)} \right) \text{ since } \pi(1) = j \\
&= \sum_{j=1}^{n} a_{1j} \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=2}^{n} a_{i\,\pi(i)} \right). \quad (*)
\end{aligned}$$

Theorem 3.1.F (continued 2)

Proof (continued). Now in $(*)$, as permutation π ranges over S_n^j and as i ranges over $\{2, 3, \ldots, n\}$, the elements $a_{i\,\pi(i)}$ ranges over all entries of A_{1j}. Since we denote the (t, s) entry of A_{1j} as $a_{ts}^{(j)}$, then we can re-index the product and inner summation in $(*)$ from $i \in \{2, 3, \ldots, n\}$ and $\pi \in S_n^j$ to $t \in \{1, 2, \ldots, n-1\}$ and $\gamma \in S_{n-1}$, respectively. We do so by defining $t = i - 1$ for $i \in \{2, 3, \ldots, n\}$ and $\gamma \in S_{n-1}$ as $\gamma(t) = \begin{cases} \pi(t+1) & \text{if } \pi(t+1) < j \\ \pi(t+1) - 1 & \text{if } \pi(t+1) > j \end{cases}$ for $t \in \{1, 2, \ldots, n-1\}$. We then have that $\gamma : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\}$ and so $\gamma \in S_{n-1}$. Also, $\gamma(t) = \pi(i)$ if $\pi(i) < j$, and $\gamma(t) = \pi(i) - 1$ if $\pi(i) > j$.

Theorem 3.1.F (continued 2)

Proof (continued). Now in $(*)$, as permutation π ranges over S_n^j and as i ranges over $\{2, 3, \ldots, n\}$, the elements $a_{i\,\pi(i)}$ ranges over all entries of A_{1j}. Since we denote the (t, s) entry of A_{1j} as $a_{ts}^{(j)}$, then we can re-index the product and inner summation in $(*)$ from $i \in \{2, 3, \ldots, n\}$ and $\pi \in S_n^j$ to $t \in \{1, 2, \ldots, n-1\}$ and $\gamma \in S_{n-1}$, respectively. We do so by defining $t = i - 1$ for $i \in \{2, 3, \ldots, n\}$ and $\gamma \in S_{n-1}$ as $\gamma(t) = \begin{cases} \pi(t+1) & \text{if } \pi(t+1) < j \\ \pi(t+1) - 1 & \text{if } \pi(t+1) > j \end{cases}$ for $t \in \{1, 2, \ldots, n-1\}$. We then have that $\gamma : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\}$ and so $\gamma \in S_{n-1}$. Also, $\gamma(t) = \pi(i)$ if $\pi(i) < j$, and $\gamma(t) = \pi(i) - 1$ if $\pi(i) > j$. Now extend $\gamma \in S_{n-1}$ to $\gamma' \in S_n$, be defining $\gamma'(t) = \gamma(t)$ for $t \in \{1, 2, \ldots, n-1\}$ and $\gamma'(n) = n$. Then $\sigma(\gamma') = \sigma(\gamma)$.

Theorem 3.1.F (continued 2)

Proof (continued). Now in $(*)$, as permutation π ranges over S_n^j and as i ranges over $\{2, 3, \ldots, n\}$, the elements $a_{i\,\pi(i)}$ ranges over all entries of A_{1j}. Since we denote the (t, s) entry of A_{1j} as $a_{ts}^{(j)}$, then we can re-index the product and inner summation in $(*)$ from $i \in \{2, 3, \ldots, n\}$ and $\pi \in S_n^j$ to $t \in \{1, 2, \ldots, n-1\}$ and $\gamma \in S_{n-1}$, respectively. We do so by defining $t = i - 1$ for $i \in \{2, 3, \ldots, n\}$ and $\gamma \in S_{n-1}$ as
$\gamma(t) = \begin{cases} \pi(t+1) & \text{if } \pi(t+1) < j \\ \pi(t+1) - 1 & \text{if } \pi(t+1) > j \end{cases}$ for $t \in \{1, 2, \ldots, n-1\}$. We then have that $\gamma : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\}$ and so $\gamma \in S_{n-1}$. Also, $\gamma(t) = \pi(i)$ if $\pi(i) < j$, and $\gamma(t) = \pi(i) - 1$ if $\pi(i) > j$. Now extend $\gamma \in S_{n-1}$ to $\gamma' \in S_n$, be defining $\gamma'(t) = \gamma(t)$ for $t \in \{1, 2, \ldots, n-1\}$ and $\gamma'(n) = n$. Then $\sigma(\gamma') = \sigma(\gamma)$.

Theorem 3.1.F (continued 3)

Proof (continued). We can relate γ' and π with the following mapping:

$$\begin{array}{ccccc} \gamma'(1) & \gamma'(2) & \cdots & \gamma'(n-1) & \gamma'(n) \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \pi(2) & \pi(3) & \cdots & \pi(n) & \pi(1)=j. \end{array}$$

We will need to "move the jth term to the right end" and do so using the mapping π'':

$$\begin{array}{ccccccccc} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ 1 & 2 & \cdots & j-1 & n & j & \cdots & n-2 & n-1 \end{array}.$$

So first we increase indices by 1 (mod n) with the permutation

$$\pi' = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} = (2,3)(3,4)\cdots(n-1,n)(n,1),$$

second we apply permutation π, and third we perform the second mapping above using the permutation π'' where ...

Theorem 3.1.F (continued 3)

Proof (continued). We can relate γ' and π with the following mapping:

$$\begin{array}{ccccc} \gamma'(1) & \gamma'(2) & \cdots & \gamma'(n-1) & \gamma'(n) \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \pi(2) & \pi(3) & \cdots & \pi(n) & \pi(1) = j. \end{array}$$

We will need to "move the jth term to the right end" and do so using the mapping π'':

$$\begin{array}{cccccccccc} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ 1 & 2 & \cdots & j-1 & n & j & \cdots & n-2 & n-1 \end{array}.$$

So first we increase indices by 1 (mod n) with the permutation

$$\pi' = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} = (2,3)(3,4)\cdots(n-1,n)(n,1),$$

second we apply permutation π, and third we perform the second mapping above using the permutation π'' where ...

Theorem 3.1.F (continued 4)

Proof (continued).

$$\pi'' = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\ 1 & 2 & \cdots & j-1 & n & j & \cdots & n-2 & n-1 \end{pmatrix}$$

$$= (n-2, n-1)(n-3, n-2)\cdots(j, j+1)(j, n).$$

Then $\gamma' = \pi''\pi\pi'$. Notice $\sigma(\pi') = (-1)^{n-1}$ and $\sigma(\pi'') = (-1)^{n-j}$, so that

$$\sigma(\gamma) = \sigma(\gamma') = \sigma(\pi''\pi\pi')$$

$$= \sigma(\pi'')\sigma(\pi)\sigma(\pi') = (-1)^{2n-j-1}\sigma(\pi) = (-1)^{j+1}\sigma(\pi),$$

or $\sigma(\pi) = (-1)^{j+1}\sigma(\gamma)$. So $(*)$ becomes

$$\det(A) = \sum_{j=1}^{n} a_{1j} \left(\sum_{\pi \in S_n^j} \sigma(\pi) \prod_{i=2}^{n} a_{i\,\pi(i)} \right) \text{ by } (*)$$

Theorem 3.1.F (continued 4)

Proof (continued).

$$\pi'' = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\ 1 & 2 & \cdots & j-1 & n & j & \cdots & n-2 & n-1 \end{pmatrix}$$

$$= (n-2, n-1)(n-3, n-2)\cdots(j, j+1)(j, n).$$

Then $\gamma' = \pi''\pi\pi'$. Notice $\sigma(\pi') = (-1)^{n-1}$ and $\sigma(\pi'') = (-1)^{n-j}$, so that

$$\sigma(\gamma) = \sigma(\gamma') = \sigma(\pi''\pi\pi')$$

$$= \sigma(\pi'')\sigma(\pi)\sigma(\pi') = (-1)^{2n-j-1}\sigma(\pi) = (-1)^{j+1}\sigma(\pi),$$

or $\sigma(\pi) = (-1)^{j+1}\sigma(\gamma)$. So $(*)$ becomes

$$\det(A) = \sum_{j=1}^{n} a_{1j}\left(\sum_{\pi\in S_n^j} \sigma(\pi)\prod_{i=2}^{n} a_{i\,\pi(i)}\right) \text{ by } (*)$$

Theorem 3.1.F (continued 5)

Proof (continued).

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} a_{1j} \left(\sum_{\gamma \in S_{n-1}} (-1)^{j+1} \sigma(\gamma) \prod_{t=1}^{n-1} a^{(j)}_{t\,\gamma(t)} \right) \\
&\quad \text{where } \gamma' = \pi''\pi\pi' \text{ and } \gamma \text{ is} \\
&\quad \text{the restriction of } \gamma' \text{ to } \{1, 2, \ldots, n-1\} \\
&= \sum_{j=1}^{n} a_{1j}(-1)^{j+1} \left(\sum_{\gamma \in S_{n-1}} \sigma(\gamma) \prod_{t=1}^{n-1} a^{(j)}_{t\,\gamma(t)} \right) \\
&= \sum_{j=1}^{n} a_{1j}(-1)^{j+1} \det(A_{1j}) = \sum_{j=1}^{n} a_{1j}\alpha_{1j},
\end{aligned}$$

and the claim holds for $i = 1$.

Theorem 3.1.F (continued 6)

Proof (continued). Consider now equation (5.1) for $i > 1$. Let B be the $n \times n$ matrix formed from A by interchanging the $(i-1)$th and ith rows, then the $(i-2)$th and $(i-1)$th rows, ..., then the 1st and 2nd rows (so that the first row of B is the ith row of A and the 2nd through ith row of B is the 1st through $(i-1)$th row of A, respectively). By Theorem 3.1.C, $\det(A) = (-1)^{i-1}\det(B)$. Let B_{1j} be the $(n-1) \times (n-1)$ matrix obtained by eliminating the 1st row and the jth column of B, and let b_{1j} be the jth element of the first row of B. Then $B_{1j} = A_{ij}$ and so

$$\begin{aligned}\det(A) &= (-1)^{i-1}\det(B) = (-1)^{i-1}\sum_{j=1}^{n} b_{1j}(-1)^{1+j}\det(B_{1j}) \\ &\quad \text{by the first part of the proof} \\ &= (-1)^{i-1}\sum_{j=1}^{n} a_{ij}(-1)^{1+j}\det(A_{ij}) \text{ since } B_{1j} = A_{ij}\end{aligned}$$

Theorem 3.1.F (continued 6)

Proof (continued). Consider now equation (5.1) for $i > 1$. Let B be the $n \times n$ matrix formed from A by interchanging the $(i-1)$th and ith rows, then the $(i-2)$th and $(i-1)$th rows, ..., then the 1st and 2nd rows (so that the first row of B is the ith row of A and the 2nd through ith row of B is the 1st through $(i-1)$th row of A, respectively). By Theorem 3.1.C, $\det(A) = (-1)^{i-1}\det(B)$. Let B_{1j} be the $(n-1) \times (n-1)$ matrix obtained by eliminating the 1st row and the jth column of B, and let b_{1j} be the jth element of the first row of B. Then $B_{1j} = A_{ij}$ and so

$$\begin{aligned} \det(A) &= (-1)^{i-1}\det(B) = (-1)^{i-1}\sum_{j=1}^{n} b_{1j}(-1)^{1+j}\det(B_{1j}) \\ &\qquad \text{by the first part of the proof} \\ &= (-1)^{i-1}\sum_{j=1}^{n} a_{ij}(-1)^{1+j}\det(A_{ij}) \text{ since } B_{1j} = A_{ij} \end{aligned}$$

Theorem 3.1.F (continued 7)

Proof (continued).

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) = \sum_{j=1}^{n} a_{ij}\alpha_{ij}.$$

So equation (5.1) holds for all $i = 1, 2, \ldots, n$.

Finally, consider equation (5.2). Notice that the matrix formed by eliminating the jth row and ith column of A^T is A_{ij}^T. So the cofactor of the (j, i)th element of A^T is $(-1)^{j+i}\det(A_{ij}^T) = (-1)^{j+i}\det(A_{ij}) = \alpha_{ij}$ by Theorem 3.1.A.

Theorem 3.1.F (continued 7)

Proof (continued).

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) = \sum_{j=1}^{n} a_{ij}\alpha_{ij}.$$

So equation (5.1) holds for all $i = 1, 2, \ldots, n$.

Finally, consider equation (5.2). Notice that the matrix formed by eliminating the jth row and ith column of A^T is A_{ij}^T. So the cofactor of the (j, i)th element of A^T is $(-1)^{j+i}\det(A_{ij}^T) = (-1)^{j+i}\det(A_{ij}) = \alpha_{ij}$ by Theorem 3.1.A. Since the (j, i)th element of A^T is the (i, j)th element of A, then by equation (5.1) and Theorem 3.1.A,

$$\begin{aligned}\det(A) &= \det(A^T) = \sum_{j=1}^{n} a'_{ij}\alpha'_{ij} \text{ where } a'_{ij} = a_{ji} \text{ and } \alpha'_{ij} = (-1)^{i+j}\det(A_{ij}^T) \\ &= \sum_{i=1}^{n} a'_{ji}\alpha'_{ji} \text{ interchanging } i \text{ and } j\end{aligned}$$

Theorem 3.1.F (continued 7)

Proof (continued).

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) = \sum_{j=1}^{n} a_{ij}\alpha_{ij}.$$

So equation (5.1) holds for all $i = 1, 2, \ldots, n$.

Finally, consider equation (5.2). Notice that the matrix formed by eliminating the jth row and ith column of A^T is A_{ij}^T. So the cofactor of the (j, i)th element of A^T is $(-1)^{j+i}\det(A_{ij}^T) = (-1)^{j+i}\det(A_{ij}) = \alpha_{ij}$ by Theorem 3.1.A. Since the (j, i)th element of A^T is the (i, j)th element of A, then by equation (5.1) and Theorem 3.1.A,

$$\begin{aligned}\det(A) &= \det(A^T) = \sum_{j=1}^{n} a'_{ij}\alpha'_{ij} \text{ where } a'_{ij} = a_{ji} \text{ and } \alpha'_{ij} = (-1)^{i+j}\det(A_{ij}^T) \\ &= \sum_{i=1}^{n} a'_{ji}\alpha'_{ji} \text{ interchanging } i \text{ and } j\end{aligned}$$

Theorem 3.1.F (continued 8)

Theorem 3.1.F. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let α_{ij} represent the cofactor of a_{ij}. Then

$$\det(A) = \sum_{i=1}^{n} a_{ij}\alpha_{ij} \text{ for } j = 1, 2, \ldots, . \qquad (5.2)$$

Proof (continued). ...

$$\begin{aligned} \det(A) &= \sum_{i=1}^{n} a_{ij}(-1)^{j+i}\det(A_{ji}^{T}) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) \\ &= \sum_{i=1}^{n} a_{ij}\alpha_{ij} \end{aligned}$$

and equation 5.2 holds. □

Theorem 3.1.3

Theorem 3.1.3. Let A be an $n \times n$ matrix with adjoint $\mathrm{adj}(A) = [\alpha_{ij}]^T$. Then $A\,\mathrm{adj}(A) = \mathrm{adj}(A)A = \det(A)I_n$.

Proof. With $A = [a_{ij}]$ we have the (i, j) entry of $A\,\mathrm{adj}(A)$ as $\sum_{k=1}^{n} a_{ik}\alpha_{jk}$. By Theorem 3.1.F, for $i = j$ this is $\det(A)$.

Theorem 3.1.3

Theorem 3.1.3. Let A be an $n \times n$ matrix with adjoint $\text{adj}(A) = [\alpha_{ij}]^T$. Then $A\,\text{adj}(A) = \text{adj}(A)A = \det(A)I_n$.

Proof. With $A = [a_{ij}]$ we have the (i,j) entry of $A\,\text{adj}(A)$ as $\sum_{k=1}^n a_{ik}\alpha_{jk}$. By Theorem 3.1.F, for $i = j$ this is $\det(A)$.
If $i \neq j$, consider the matrix $B = [b_{ij}]$ where B is $n \times n$ and has the same rows as A, except that its jth row is the same as the ith row of A. Then the cofactors α_{jk} of A are the same as the cofactors β_{jk} of B for $1 \leq k \leq n$. Also, since the jth row of B is the same as the ith of A then $b_{jk} = a_{ik}$ for $1 \leq k \leq n$. Since the ith row and the jth row are the same in B then, by Note 3.1.C, $\det(B) = 0$.

Theorem 3.1.3

Theorem 3.1.3. Let A be an $n \times n$ matrix with adjoint $\mathrm{adj}(A) = [\alpha_{ij}]^T$. Then $A\,\mathrm{adj}(A) = \mathrm{adj}(A)A = \det(A)I_n$.

Proof. With $A = [a_{ij}]$ we have the (i,j) entry of $A\,\mathrm{adj}(A)$ as $\sum_{k=1}^n a_{ik}\alpha_{jk}$. By Theorem 3.1.F, for $i = j$ this is $\det(A)$.
If $i \neq j$, consider the matrix $B = [b_{ij}]$ where B is $n \times n$ and has the same rows as A, except that its jth row is the same as the ith row of A. Then the cofactors α_{jk} of A are the same as the cofactors β_{jk} of B for $1 \le k \le n$. Also, since the jth row of B is the same as the ith of A then $b_{jk} = a_{ik}$ for $1 \le k \le n$. Since the ith row and the jth row are the same in B then, by Note 3.1.C, $\det(B) = 0$. So for $i \neq j$ the (i,j) entry of $A\,\mathrm{adj}(A)$ is

$$\sum_{k=1}^n a_{ik}\alpha_{jk} = \sum_{k=1}^n b_{jk}\beta_{jk} = \det(B) = 0 \text{ by Theorem 3.1.F.}$$

So the (i,j) entry of $A\,\mathrm{adj}(A)$ is $\det(A)$ for $i = j$ and 0 for $i \neq j$; that is $A\,\mathrm{adj}(A) = \det(A)I_n$, as claimed. Similarly, $\mathrm{adj}(A)A = \det(A)I_n$. □

Theorem 3.1.3

Theorem 3.1.3. Let A be an $n \times n$ matrix with adjoint $\text{adj}(A) = [\alpha_{ij}]^T$. Then $A\,\text{adj}(A) = \text{adj}(A)A = \det(A)I_n$.

Proof. With $A = [a_{ij}]$ we have the (i,j) entry of $A\,\text{adj}(A)$ as $\sum_{k=1}^n a_{ik}\alpha_{jk}$. By Theorem 3.1.F, for $i = j$ this is $\det(A)$.
If $i \neq j$, consider the matrix $B = [b_{ij}]$ where B is $n \times n$ and has the same rows as A, except that its jth row is the same as the ith row of A. Then the cofactors α_{jk} of A are the same as the cofactors β_{jk} of B for $1 \leq k \leq n$. Also, since the jth row of B is the same as the ith of A then $b_{jk} = a_{ik}$ for $1 \leq k \leq n$. Since the ith row and the jth row are the same in B then, by Note 3.1.C, $\det(B) = 0$. So for $i \neq j$ the (i,j) entry of $A\,\text{adj}(A)$ is

$$\sum_{k=1}^n a_{ik}\alpha_{jk} = \sum_{k=1}^n b_{jk}\beta_{jk} = \det(B) = 0 \text{ by Theorem 3.1.F.}$$

So the (i,j) entry of $A\,\text{adj}(A)$ is $\det(A)$ for $i = j$ and 0 for $i \neq j$; that is $A\,\text{adj}(A) = \det(A)I_n$, as claimed. Similarly, $\text{adj}(A)A = \det(A)I_n$. □

Theorem 3.1.G

Theorem 3.1.G. Let T be an $m \times m$ matrix, V an $n \times m$ matrix, W an $n \times n$ matrix, and let '0' represent the $m \times n$ matrix of all entries as 0. Then the determinant of the partitioned matrix is

$$\det \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = \det \begin{bmatrix} W & V \\ 0 & T \end{bmatrix} = \det(T)\det(W).$$

Proof. Let $A = \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = [a_{ij}]$ be a partitioned $(m+n) \times (m+n)$ matrix. Let $T = [t_{ij}]$ and $W = [w_{ij}]$, so that $t_{ij} = a_{ij}$ for $i, j \in \{1, 2, \ldots, m\}$ and $w_{ij} = a_{(i+m)\,(j+m)}$ for $i, j \in \{1, 2, \ldots, n\}$. By definition: $\det(A) = \sum_{\pi \in S_{m+n}} \sigma(\pi) \prod_{i=1}^{m+n} a_{i\,\pi(i)}$. $\quad (*)$

Theorem 3.1.G

Theorem 3.1.G. Let T be an $m \times m$ matrix, V an $n \times m$ matrix, W an $n \times n$ matrix, and let '0' represent the $m \times n$ matrix of all entries as 0. Then the determinant of the partitioned matrix is

$$\det \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = \det \begin{bmatrix} W & V \\ 0 & T \end{bmatrix} = \det(T)\det(W).$$

Proof. Let $A = \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = [a_{ij}]$ be a partitioned $(m+n) \times (m+n)$ matrix. Let $T = [t_{ij}]$ and $W = [w_{ij}]$, so that $t_{ij} = a_{ij}$ for $i, j \in \{1, 2, \ldots, m\}$ and $w_{ij} = a_{(i+m)\,(j+m)}$ for $i, j \in \{1, 2, \ldots, n\}$. By definition: $\det(A) = \sum_{\pi \in S_{m+n}} \sigma(\pi) \prod_{i=1}^{m+n} a_{i\,\pi(i)}$. $\qquad (*)$

Now the only time the product in $(*)$ *might* be nonzero is when π is a permutation mapping $\{1, 2, \ldots, m\}$ to itself (otherwise $a_{i\,\pi(i)} = 0$ for some $i \in \{1, 2, \ldots, m\}$), and hence also mapping $\{m+1, m+2, \ldots, m+n\}$ to itself. Denote all such permutations as S'_{m+n}.

Theorem 3.1.G

Theorem 3.1.G. Let T be an $m \times m$ matrix, V an $n \times m$ matrix, W an $n \times n$ matrix, and let '0' represent the $m \times n$ matrix of all entries as 0. Then the determinant of the partitioned matrix is

$$\det \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = \det \begin{bmatrix} W & V \\ 0 & T \end{bmatrix} = \det(T)\det(W).$$

Proof. Let $A = \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} = [a_{ij}]$ be a partitioned $(m+n) \times (m+n)$ matrix. Let $T = [t_{ij}]$ and $W = [w_{ij}]$, so that $t_{ij} = a_{ij}$ for $i, j \in \{1, 2, \ldots, m\}$ and $w_{ij} = a_{(i+m)\,(j+m)}$ for $i, j \in \{1, 2, \ldots, n\}$. By definition: $\det(A) = \sum_{\pi \in S_{m+n}} \sigma(\pi) \prod_{i=1}^{m+n} a_{i\,\pi(i)}$. $\quad (*)$

Now the only time the product in $(*)$ *might* be nonzero is when π is a permutation mapping $\{1, 2, \ldots, m\}$ to itself (otherwise $a_{i\,\pi(i)} = 0$ for some $i \in \{1, 2, \ldots, m\}$), and hence also mapping $\{m+1, m+2, \ldots, m+n\}$ to itself. Denote all such permutations as S'_{m+n}.

Theorem 3.1.G (continued 1)

Proof (continued). Such $\pi \in S'_{m+n}$ can be written as the product of two permutations, π_m and π_n, in S'_{m+n} where π_m fixes $\{m+1, m+2, \ldots, m+n\}$ and π_n fixes $\{1, 2, \ldots, m\}$; that is, $\pi = \pi_m \pi_n$ and $\sigma(\pi) = \sigma(\pi_m)\sigma(\pi_n)$. Now if we restrict π_m to $\{1, 2, \ldots, m\}$ and denote the resulting function as π'_m then we have $\pi'_m \in S_m$. If we define $\pi'_n(i - m) = \pi_n(i) - m$ for $i \in \{m+1, m+2, \ldots, m+n\}$, then $\pi'_n : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and $\pi'_n \in S_n$. We have $\sigma(\pi_m) = \sigma(\pi'_m)$ and $\sigma(\pi_n) = \sigma(\pi'_n)$.

Theorem 3.1.G (continued 1)

Proof (continued). Such $\pi \in S'_{m+n}$ can be written as the product of two permutations, π_m and π_n, in S'_{m+n} where π_m fixes $\{m+1, m+2, \ldots, m+n\}$ and π_n fixes $\{1, 2, \ldots, m\}$; that is, $\pi = \pi_m \pi_n$ and $\sigma(\pi) = \sigma(\pi_m)\sigma(\pi_n)$. Now if we restrict π_m to $\{1, 2, \ldots, m\}$ and denote the resulting function as π'_m then we have $\pi'_m \in S_m$. If we define $\pi'_n(i-m) = \pi_n(i) - m$ for $i \in \{m+1, m+2, \ldots, m+n\}$, then $\pi'_n : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and $\pi'_n \in S_n$. We have $\sigma(\pi_m) = \sigma(\pi'_m)$ and $\sigma(\pi_n) = \sigma(\pi'_n)$. So from $(*)$ we have

$$\begin{aligned}
\det(A) &= \sum_{\pi \in S'_{m+n}} \sigma(\pi) \prod_{i=1}^{m+n} a_{i\,\pi(i)} \\
&= \sum_{\pi_m, \pi_n \in S'_{m+n}} \sigma(\pi_m)\sigma(\pi_n) \prod_{i=1}^{m} a_{i\,\pi_m(i)} \prod_{i=m+1}^{m+n} a_{i\,\pi_n(i)}
\end{aligned}$$

where each $\pi \in S'_{m+n}$ is written as $\pi = \pi_m \pi_n$

Theorem 3.1.G (continued 1)

Proof (continued). Such $\pi \in S'_{m+n}$ can be written as the product of two permutations, π_m and π_n, in S'_{m+n} where π_m fixes $\{m+1, m+2, \ldots, m+n\}$ and π_n fixes $\{1, 2, \ldots, m\}$; that is, $\pi = \pi_m \pi_n$ and $\sigma(\pi) = \sigma(\pi_m)\sigma(\pi_n)$. Now if we restrict π_m to $\{1, 2, \ldots, m\}$ and denote the resulting function as π'_m then we have $\pi'_m \in S_m$. If we define $\pi'_n(i-m) = \pi_n(i) - m$ for $i \in \{m+1, m+2, \ldots, m+n\}$, then $\pi'_n : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and $\pi'_n \in S_n$. We have $\sigma(\pi_m) = \sigma(\pi'_m)$ and $\sigma(\pi_n) = \sigma(\pi'_n)$. So from $(*)$ we have

$$\begin{aligned} \det(A) &= \sum_{\pi \in S'_{m+n}} \sigma(\pi) \prod_{i=1}^{m+n} a_{i\,\pi(i)} \\ &= \sum_{\pi_m, \pi_n \in S'_{m+n}} \sigma(\pi_m)\sigma(\pi_n) \prod_{i=1}^{m} a_{i\,\pi_m(i)} \prod_{i=m+1}^{m+n} a_{i\,\pi_n(i)} \end{aligned}$$

where each $\pi \in S'_{m+n}$ is written as $\pi = \pi_m \pi_n$

Theorem 3.1.G (continued 2)

Proof (continued).

$$\begin{aligned}
\det(A) &= \sum_{\pi_m,\pi_n\in S'_{m+n}} \sigma(\pi_m)\sigma(\pi_n)\prod_{i=1}^{m} a_{i\,\pi_m(i)} \prod_{i=m+1}^{m+n} a_{i\,\pi_n(i)} \\
&= \sum_{\pi'_m\in S_m,\pi'_n\in S_n} \sigma(\pi'_m)\sigma(\pi'_n)\prod_{i=1}^{m} a_{i\,\pi'_m(i)} \prod_{i=1}^{n} a_{(i+m)\,\pi_n(i+m)} \\
&= \sum_{\pi'_m\in S_m,\pi'_n\in S_n} \sigma(\pi'_m)\sigma(\pi'_n)\prod_{i=1}^{m} a_{i\,\pi'_m(i)} \prod_{i=1}^{n} a_{(i+m)\,\pi'_n(i)+m}
\end{aligned}$$

since $\pi'_n(i-m)=\pi_n(i)-m$ for $i\in\{m+1,m+2,\ldots,m+n\}$
or $\pi'_n(i)+m=\pi_n(i+m)$ for $i\in\{1,2,\ldots,n\}$

$$= \sum_{\pi'_m\in S_m,\pi'_n\in S_n} \sigma(\pi'_m)\sigma(\pi'_n)\prod_{i=1}^{m} t_{i\,\pi'_m(i)} \prod_{i=1}^{n} w_{i\,\pi'_n(i)}$$

Theorem 3.1.G (continued 3)

Proof (continued).

$$\begin{aligned}\det(A) &= \sum_{\pi'_m \in S_m, \pi'_n \in S_n} \sigma(\pi'_m)\sigma(\pi'_n) \prod_{i=1}^{m} t_{i\,\pi'_m(i)} \prod_{i=1}^{n} w_{i\,\pi'_n(i)} \\ &= \sum_{\pi'_m \in S_m} \sigma(\pi'_m) \prod_{i=1}^{m} t_{i\,\pi'_m(i)} \sum_{\pi'_n \in S_n} \sigma(\pi'_n) \prod_{i=1}^{n} w_{i\,\pi'_n(i)} \\ &= \det(T)\det(W).\end{aligned}$$

The proof that $\det \begin{bmatrix} W & V \\ 0 & T \end{bmatrix} = \det(T)\det(W)$ is similar. □

Theorem 3.1.G (continued 3)

Proof (continued).

$$\begin{aligned}\det(A) &= \sum_{\pi'_m \in S_m, \pi'_n \in S_n} \sigma(\pi'_m)\sigma(\pi'_n) \prod_{i=1}^{m} t_{i\,\pi'_m(i)} \prod_{i=1}^{n} w_{i\,\pi'_n(i)} \\ &= \sum_{\pi'_m \in S_m} \sigma(\pi'_m) \prod_{i=1}^{m} t_{i\,\pi'_m(i)} \sum_{\pi'_n \in S_n} \sigma(\pi'_n) \prod_{i=1}^{n} w_{i\,\pi'_n(i)} \\ &= \det(T)\det(W).\end{aligned}$$

The proof that $\det \begin{bmatrix} W & V \\ 0 & T \end{bmatrix} = \det(T)\det(W)$ is similar. □

Theorem 3.1.H

Theorem 3.1.H. Let A be $n \times n$ and let T be an $n \times n$ upper or lower triangular matrix with entries of 1 along the diagonal. Then $\det(AT) = \det(TA) = \det(A)$.

Proof. Consider the case AT where T is lower triangular. Define T_i to be an $n \times n$ matrix formed from I_n by replacing the ith column of I_n with the ith column of T (for $1 \leq i \leq n$). Then $T = T_1 T_2 \cdots T_n$, as shown in Exercise 3.1.C, so $AT = AT_1 T_2 \cdots T_n$.

Theorem 3.1.H

Theorem 3.1.H. Let A be $n \times n$ and let T be an $n \times n$ upper or lower triangular matrix with entries of 1 along the diagonal. Then $\det(AT) = \det(TA) = \det(A)$.

Proof. Consider the case AT where T is lower triangular. Define T_i to be an $n \times n$ matrix formed from I_n by replacing the ith column of I_n with the ith column of T (for $1 \le i \le n$). Then $T = T_1 T_2 \cdots T_n$, as shown in Exercise 3.1.C, so $AT = AT_1 T_2 \cdots T_n$. Define $B_0 = A$ and $B_i = AT_1 T_2 \cdots T_i$ (for $1 \le i \le n$). Consider $B_{i-1} T_i$ for $1 \le i \le n$. Since all columns of T_i, except for the ith column, are the same as I_n then the columns of $B_{i-1} T_i$ are the same as the columns of B_{i-1}, except for the ith column. Let $t_{1i}, t_{2i}, \ldots, t_{ni}$ be the entries in the ith column of T_i (so $t_{1i} = t_{2i} = \cdots = t_{(i-1)i} = 0$ and $t_{ii} = 1$). Let $b_1, b_2, \ldots, b_n$ be the columns of B_{i-1}.

Theorem 3.1.H

Theorem 3.1.H. Let A be $n \times n$ and let T be an $n \times n$ upper or lower triangular matrix with entries of 1 along the diagonal. Then $\det(AT) = \det(TA) = \det(A)$.

Proof. Consider the case AT where T is lower triangular. Define T_i to be an $n \times n$ matrix formed from I_n by replacing the ith column of I_n with the ith column of T (for $1 \le i \le n$). Then $T = T_1 T_2 \cdots T_n$, as shown in Exercise 3.1.C, so $AT = AT_1 T_2 \cdots T_n$. Define $B_0 = A$ and $B_i = AT_1 T_2 \cdots T_i$ (for $1 \le i \le n$). Consider $B_{i-1} T_i$ for $1 \le i \le n$. Since all columns of T_i, except for the ith column, are the same as I_n then the columns of $B_{i-1} T_i$ are the same as the columns of B_{i-1}, except for the ith column. Let $t_{1i}, t_{2i}, \ldots, t_{ni}$ be the entries in the ith column of T_i (so $t_{1i} = t_{2i} = \cdots = t_{(i-1)i} = 0$ and $t_{ii} = 1$). Let $b_1, b_2, \ldots, b_n$ be the columns of B_{i-1}.

Theorem 3.1.H (continued)

Theorem 3.1.H. Let A be $n \times n$ and let T be an $n \times n$ upper or lower triangular matrix with entries of 1 along the diagonal. Then $\det(AT) = \det(TA) = \det(A)$.

Proof (continued). Then the entries of the ith column of $B_{i-1}T_i$ are

$$\sum_{k=1}^{n} b_{jk}t_{ki} = b_{ji} + \sum_{k=i+1}^{n} b_{jk}t_{ki} \text{ for } 1 \leq j \leq n$$

where the entries of b_i are $b_{1i}, b_{2i}, \ldots, b_{ni}$. So the ith column of $B_{i-1}T_i$ is $b_i + \sum_{k=i+1}^{n} b_k t_{ki}$, which is the ith column of B_{i-1} plus a series of scalar multiples of the columns $b_{i+1}, b_{i+2}, \ldots, b_n$ of B_{i-1}. So by Theorem 3.1.E, $\det(B_i) = \det(B_{i-1}T_i) = \det(B_{i-1})$. This holds for $1 \leq i \leq n$, so

$$\det(A) = \det(B_0) = \det(B_1) = \det(B_2) = \cdots = \det(B_n) = \det(AT).$$

The result holds similarly for T upper triangular and for TA. □

Theorem 3.1.H (continued)

Theorem 3.1.H. Let A be $n \times n$ and let T be an $n \times n$ upper or lower triangular matrix with entries of 1 along the diagonal. Then $\det(AT) = \det(TA) = \det(A)$.

Proof (continued). Then the entries of the ith column of $B_{i-1}T_i$ are

$$\sum_{k=1}^{n} b_{jk}t_{ki} = b_{ji} + \sum_{k=i+1}^{n} b_{jk}t_{ki} \text{ for } 1 \leq j \leq n$$

where the entries of b_i are $b_{1i}, b_{2i}, \ldots, b_{ni}$. So the ith column of $B_{i-1}T_i$ is $b_i + \sum_{k=i+1}^{n} b_k t_{ki}$, which is the ith column of B_{i-1} plus a series of scalar multiples of the columns $b_{i+1}, b_{i+2}, \ldots, b_n$ of B_{i-1}. So by Theorem 3.1.E, $\det(B_i) = \det(B_{i-1}T_i) = \det(B_{i-1})$. This holds for $1 \leq i \leq n$, so

$$\det(A) = \det(B_0) = \det(B_1) = \det(B_2) = \cdots = \det(B_n) = \det(AT).$$

The result holds similarly for T upper triangular and for TA. □