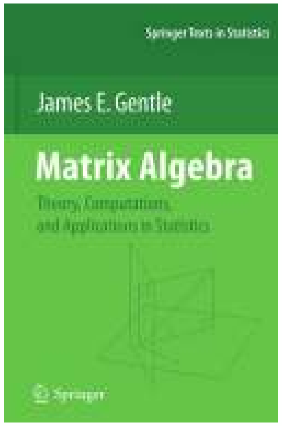
Theory of Matrices

Chapter 3. Basic Properties of Matrices

3.3. Matrix Rank and the Inverse of a Full Rank Matrix—Proofs of Theorems



Lemma 3.3.1

Lemma 3.3.1. Let $\{a^i\}_{i=1}^k = \{[a_1^i, a_2^i, \ldots, a_n^i]\}_{i=1}^k$ be a set of vectors in $\mathbb{R}^n$ and let $\pi \in S_n$. Then the set of vectors $\{a^i\}_{i=1}^k$ is linearly independent if and only if the set of vectors $\{[a_{\pi(1)}^i, a_{\pi(2)}^i, \ldots, a_{\pi(n)}^i]\}_{i=1}^k$ is linearly independent. That is, permuting all the entries in a set of vectors by the same permutation preserves the linear dependence/independence of the set.

Proof. Set $\{a^i\}_{i=1}^k$ is linearly independent if and only if $\sum_{i=1}^k s_i a^i = 0$ for scalars $s_1, s_2, \ldots, s_k$ implies $s_1 = s_2 = \cdots = s_k = 0$. Now $\sum_{i=1}^k s_i a^i = 0$ implies that $\sum_{i=1}^k s_i a_j^i = 0$ for $j = 1, 2, \ldots, n$. So this system of n linear equations (in k unknowns s_i for $i = 1, 2, \ldots, k$) has only one solution if and only if the system of n linear equations in k unknowns $\sum_{i=1}^k s_i a_{\pi(j)}^i = 0$ for $j = 1, 2, \ldots, n$ has only one solution, namely $s_1 = s_2 = \cdots = s_k = 0$. That is, if and only if the vector equation $\sum_{i=1}^k s_i b^i = 0$, where $b^i = [a_{\pi(1)}^i, a_{\pi(2)}^i, \ldots, a_{\pi(n)}^i]$ for $i = 1, 2, \ldots, k$, has only one solution, namely $s_1 = s_2 = \cdots s_k = 0$.

Lemma 3.3.1 (continued)

Lemma 3.3.1. Let $\{a^i\}_{i=1}^k = \{[a_1^i, a_2^i, \ldots, a_n^i]\}_{i=1}^k$ be a set of vectors in $\mathbb{R}^n$ and let $\pi \in S_n$. Then the set of vectors $\{a^i\}_{i=1}^k$ is linearly independent if and only if the set of vectors $\{[a_{\pi(1)}^i, a_{\pi(2)}^i, \ldots, a_{\pi(n)}^i]\}_{i=1}^k$ is linearly independent. That is, permuting all the entries in a set of vectors by the same permutation preserves the linear dependence/independence of the set.

Proof (continued). So the set of vectors $\{b^i\}_{i=1}^k = \{[a_{\pi(1)}^i, a_{\pi(2)}^i, \ldots, a_{\pi(n)}^i]\}_{i=1}^k$ is linearly independent as well. Similarly, if $\{a^i\}$ is linearly dependent then $\{b^i\}$ is linearly dependent. □

Theorem 3.3.2

Theorem 3.3.2. Let A be an $n \times m$ matrix. Then the row rank of A equals the column rank of A. This common quantity is called the *rank* of A.

Proof. Let the row rank of A be p and let the column rank of A be q. Rearrange the rows of A to form matrix B so that the first p rows of matrix B are linearly independent (so $B = PA$ where P is some permutation matrix). Since A and B have the same rows, they have equal row rank. By Lemma 3.3.1, the column rank of A equals the column rank of B (by interchanging row i and j of A, we are interchanging all of the ith entries with the jth entries in the column vectors of A). So we can partition B as $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ where the p rows of B_1 are linearly independent and the $n - p$ rows of B_2 are (each) linear combinations of the rows of B_1. So with the rows of B_1 as $r_1, r_2, \ldots, r_p$ and the rows of B_2 as $r_{p+1}, r_{p+2}, \ldots, r_n$, we have scalars $s_{\ell i}$ where $r_\ell = \sum_{i=1}^p s_{\ell i} r_i$ for $\ell = p+1, p+2, \ldots, n$.

Theorem 3.3.2 (continued)

Proof (continued). Then with S the $(n-p)\times p$ matrix with entries $s_{\ell i}$, $S=[s_{\ell i}]$, we have $B_2 = SB_1$. So $B = \begin{bmatrix} B_1 \\ SB_1 \end{bmatrix}$. We claim now that the column rank of B is the same as the column rank of B_1.

With $s=[s_1,s_2,\ldots,s_m]^T$ as a vector of m scalars, we have $Bs=0$ if and only if $\begin{bmatrix} B_1 \\ SB_1 \end{bmatrix} s = \begin{bmatrix} B_1 s \\ SB_1 s \end{bmatrix} = 0$ if and only if $B_1 s = 0$. That is, a linear combination of the columns of B is 0 if and only if the corresponding linear combination of the columns of B_1 is 0. So the column rank of B is the same as the column rank of B_1, and so both are the same as the column rank of A (namely, q). Since the columns of B_1 are vectors in $\mathbb{R}^p$ then $q \le p$.

Similarly, we can rearrange the columns of A and partition the resulting matrix to show that $p \le q$. Therefore the row rank, p, of matrix A equals the column rank, q, of matrix A. □

Theorem 3.3.3

Theorem 3.3.3. If P and Q are products of elementary matrices then $\text{rank}(PAQ) = \text{rank}(A)$.

Proof. We show the result holds for P a single elementary matrix. The result for Q a single elementary matrix follows similarly and the general result then follows by induction. Let $P = E_{pq}$ where $I_n \overset{R_q \leftrightarrow R_p}{\sim} E_{pq}$. Then $E_{pq}A$ has the same rows as A and so $\text{rank}(E_{pq}A) = \text{rank}(A)$. Let $P = E_{sp}$ where $I_n \overset{R_p \to sR_p}{\sim} E_{sp}$ where $s \ne 0$. Then with $r_1, r_2, \ldots, r_n$ as the rows of A, we have that $r_1, r_2, \ldots, r_{p-1}, sr_p, r_{p+1}, \ldots, r_n$ are the rows of $E_{sp}A$. Now

$$\sum_{i=1}^{n} s_i r_i = \sum_{i=1}^{p-1} s_i r_i + (s_p/s)(sr_p) + \sum_{i=p+1}^{n} s_i r_i$$

for any scalars $s_1, s_2, \ldots, s_n$. So $r_1, r_2, \ldots, r_n$ and $r_1, r_2, \ldots, r_{p-1}, sr_p, r_{p+1}, \ldots, r_n$ satisfy precisely the same dependence/independence relations. Therefore $\text{rank}(E_{sp}A) = \text{rank}(A)$.

Theorem 3.3.3 (continued)

Theorem 3.3.3. If P and Q are products of elementary matrices then $\text{rank}(PAQ) = \text{rank}(A)$.

Proof (continued). Let $P = E_{psq}$ where $I_n \overset{R_p \to R_p + sR_q}{\sim} E_{psq}$. Then for $r_1, r_2, \ldots, r_n$ the rows of A, we have that $r_1, r_2, \ldots, r_{p-1}, r_p + sr_q, r_{p+1}, \ldots, r_n$ are the rows of $E_{psq}A$. Now

$$\sum_{i=1}^{p-1} s_i r_i + s_p(r_p + sr_q) + \sum_{i=p+1}^{n} s_i r_i = \sum_{i=1}^{q-1} s_i r_i + (s_p s + s_q) r_q + \sum_{i=q+1}^{n} s_i r_i$$

for any scalars $s_1, s_2, \ldots, s_n$. So $r_1, r_2, \ldots, r_n$ and $r_1, r_2, \ldots, r_{p-1}, r_p + sr_q, r_{p+1}, \ldots, r_n$ satisfy precisely the same dependence/independence relations. Therefore $\text{rank}(E_{psq}A) = \text{rank}(A)$. □

Theorem 3.3.4

Theorem 3.3.4. Let A be a matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Then

(i) $\text{rank}(A_{ij}) \le \text{rank}(A)$ for $i, j \in \{1, 2\}$.

(ii) $\text{rank}(A) \le \text{rank}([A_{11}|A_{12}]) + \text{rank}([A_{21}|A_{22}])$.

(iii) $\text{rank}(A) \le \text{rank}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$.

(iv) If $\mathcal{V}([A_{11}|A_{12}]^T) \perp \mathcal{V}([A_{21}|A_{22}]^T)$ then $\text{rank}(A) = \text{rank}([A_{11}|A_{12}]) + \text{rank}([A_{21}|A_{22}])$ and if $\mathcal{V}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) \perp \mathcal{V}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$ then

$$\text{rank}(A) = \text{rank}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right).$$

Theorem 3.3.4 (continued 1)

(i) $\text{rank}(A_{ij}) \leq \text{rank}(A)$ for $i, j \in \{1, 2\}$.

Proof. (i) Since the set of rows of $[A_{11}|A_{12}]$ is a subset of the set of rows of A, then by Exercise 2.1.G(i), $\text{rank}([A_{11}|A_{12}]) \leq \text{rank}(A)$. Similarly, the set of columns of $\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$ is a subset of the set of columns of A and so $\text{rank}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) \leq \text{rank}(A)$. Also, $\text{rank}([A_{21}|A_{22}]) \leq \text{rank}(A)$ and $\text{rank}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right) \leq \text{rank}(A)$. Next, the set of columns of A_{11} is a subset of the set of columns of $[A_{11}|A_{12}]$ and so $\text{rank}(A_{11}) \leq \text{rank}([A_{11}|A_{12}])$ (and similarly $\text{rank}(A_{12}) \leq \text{rank}([A_{11}|A_{12}])$). Therefore $\text{rank}(A_{11}) \leq \text{rank}(A_{11}|A_{12}]) \leq \text{rank}(A)$ and $\text{rank}(A_{12}) \leq \text{rank}(A_{11}|A_{12}])$ $\leq \text{rank}(A)$. Similarly, $\text{rank}(A_{21}) \leq \text{rank}(A_{21}|A_{22}]) \leq \text{rank}(A)$ and $\text{rank}(A_{22}) \leq \text{rank}(A_{21}|A_{22}]) \leq \text{rank}(A)$.

Theorem 3.3.4 (continued 2)

(ii) $\text{rank}(A) \leq \text{rank}([A_{11}|A_{12}]) + \text{rank}([A_{21}|A_{22}])$.

(iii) $\text{rank}(A) \leq \text{rank}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$.

Proof (continued). (ii) Let R be the set of rows of A, R_1 the set of rows of $[A_{11}|A_{12}]$, and R_2 the set of rows of $[A_{21}|A_{22}]$. Then $R = R_1 \cup R_2$ and by Exercise 2.1.G(ii), $\dim(\text{span}(R)) \leq \dim(\text{span}(R_1)) + \dim(\text{span}(R_2))$. That is, $\text{rank}(A) \leq \text{rank}([A_{11}|A_{12}]) + \text{rank}([A_{21}|A_{22}])$.

(iii) Let C be the set of columns of A, C_1 be the set of columns of $\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$, and C_2 be the set of columns of $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$. Then $C = C_1 \cup C_2$ and by Exercise 2.1.G(ii), $\dim(\text{span}(C)) \leq \dim(\text{span}(C_1)) + \dim(\text{span}(C_2))$. That is,

$$\text{rank}(A) \leq \text{rank}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right).$$

Theorem 3.3.4 (continued 3)

(iv) If $\mathcal{V}([A_{11}|A_{12}]^T) \perp \mathcal{V}([A_{21}|A_{22}]^T)$ then

$$\text{rank}(A) = \text{rank}([A_{11}|A_{12}]) + \text{rank}([A_{21}|A_{22}])$$

and if $\mathcal{V}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) \perp \mathcal{V}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$ then

$$\text{rank}(A) = \text{rank}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right).$$

Proof (continued). (iv) Let R be the set of rows of A, R_1 the set of rows of $[A_{11}|A_{12}]$, and R_2 the set of rows of $[A_{21}|A_{22}]$. Then $\mathcal{V}([A_{11}|A_{12}]^T)$ is the row space of $[A_{11}|A_{12}]$ and $\mathcal{V}([A_{21}|A_{22}]^T)$ is the row space of $[A_{21}|A_{22}]$. So the row space of A is $\mathcal{V}([A_{11}|A_{12}]^T) + \mathcal{V}(A_{21}|A_{22}]^T)$ (see page 13 of the text). Since $\mathcal{V}([A_{21}|A_{22}]^T) \perp \mathcal{V}([A_{21}|A_{22}]^T)$ by hypothesis, then the row space of A is $\mathcal{V}([A_{11}|A_{12}]^T) \oplus \mathcal{V}([A_{21}|A_{22}])$. By Exercise 2.1.G(iii), $\text{rank}(A) = \dim(\mathcal{V}([A_{11}|A_{12}]^T)) + \dim(\mathcal{V}([A_{21}|A_{22}]^T))$ $= \text{rank}([A_{11}|A_{12}]) + \text{rank}([A_{11}|A_{12}])$.

Theorem 3.3.4 (continued 4)

Proof (continued). (iv) Let C be the set of columns of A, C_1 the set of columns of $\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$, and C_2 the set of columns of $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$. Then $\mathcal{V}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right)$ is the column space of $\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$ and $\mathcal{V}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$ is the column space of $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$. So the columns space of A is $\mathcal{V}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) + \mathcal{V}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$. Since $\mathcal{V}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) \perp \mathcal{V}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$ by hypothesis, then the column space of A is $\mathcal{V}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) \oplus \mathcal{V}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)$.

By Exercise 2.1.G(iii),

$\text{rank}(A) = \dim\left(\mathcal{V}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right)\right) + \dim\left(\mathcal{V}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right)\right) =$

$\text{rank}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right).$ □

Theorem 3.3.5

Theorem 3.3.5. Let A be an $n \times k$ matrix and B be a $k \times m$ matrix. Then $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$.

Proof. Let the columns of A be $a_1, a_2, \ldots, a_k$, the columns of B be $b_1, b_2, \ldots, b_m$, and the columns of AB be $c_1, c_2, \ldots, c_m$. Recall (see the note on page 5 of the class notes for Section 3.2) that if $x \in \mathbb{R}^k$ then Ax is a linear combination of the columns of A; that is, $Ax \in \mathcal{V}(A)$. Now from the definition of matrix multiplication, we have $c_i = Ab_i$ for $i = 1, 2, \ldots, m$ so that $c_i = Ab_i \in \mathcal{V}(A)$ for $i = 1, 2, \ldots, m$. So every linear combination of the columns of AB is also a linear combination of the columns of A, and $\mathcal{V}(AB)$ is a subspace of $\mathcal{V}(A)$. Hence $\text{rank}(AB) \le \text{rank}(A)$. By Theorem 3.3.2, $\text{rank}(A) = \text{rank}(A^T)$, $\text{rank}(B) = \text{rank}(B^T)$, and $\text{rank}(AB) = \text{rank}((AB)^T)$. So the previous argument shows that

$$\text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^TA^T) \le \text{rank}(B^T) = \text{rank}(B).$$

Therefore, $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$. □

Theorem 3.3.6

Theorem 3.3.6. Let A and B be $n \times m$ matrices. Then

$$|\text{rank}(A) - \text{rank}(B)| \le \text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B).$$

Proof. By Theorem 3.2.2 we have

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ I_m & 0 \end{bmatrix} = \begin{bmatrix} AI_m + BI_m & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}$$

(or, eliminating the 0 matrices as Gentle does, $[A \mid B] \begin{bmatrix} I_m \\ I_m \end{bmatrix} = A + B$).

So by Theorem 3.3.5,

$$\text{rank}\left(\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}\right) \le \min\left\{\text{rank}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right), \text{rank}\left(\begin{bmatrix} I_m & 0 \\ I_m & 0 \end{bmatrix}\right)\right\}$$
$$\le \text{rank}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right).$$

Theorem 3.3.6 (continued 1)

Proof (continued). By Theorem 3.3.4(iii),

$$\text{rank}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \le \text{rank}\left(\begin{bmatrix} A \\ 0 \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} B \\ 0 \end{bmatrix}\right)$$

and so, combining these last two results,

$$\text{rank}\left(\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}\right) \le \text{rank}\left(\begin{bmatrix} A \\ 0 \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} B \\ 0 \end{bmatrix}\right).$$

Now the 0 matrices in the second rows of these matrices do not effect ranks. That is, $\text{rank}\left(\begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{rank}([A+B \mid 0])$, $\text{rank}\left(\begin{bmatrix} A \\ 0 \end{bmatrix}\right) = \text{rank}(A)$, and $\text{rank}\left(\begin{bmatrix} B \\ 0 \end{bmatrix}\right) = \text{rank}(B)$ (this can be justified by Theorem 3.3.4(iv) since $\text{rank}(0) = 0$). Similarly, $\text{rank}([A+B \mid 0]) = \text{rank}(A+B)$. Therefore,

$$\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B). \qquad (*)$$

Theorem 3.3.6 (continued 2)

Theorem 3.3.6. Let A and B be $n \times m$ matrices. Then

$$|\text{rank}(A) - \text{rank}(B)| \le \text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B).$$

Proof (continued). With the second inequality established, we have

$$\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B). \qquad (*)$$

Next, $A = (A+B) - B$, so by $(*)$ we have

$$\text{rank}(A) = \text{rank}((A+B) - B) \le \text{rank}(A+B) + \text{rank}(-B)$$

or

$$\text{rank}(A+B) \ge \text{rank}(A) - \text{rank}(-B) = \text{rank}(A) - \text{rank}(B)$$

since $\text{rank}(-B) = \text{rank}(B)$. Similarly (interchanging A and B), $\text{rank}(A+B) \ge \text{rank}(B) - \text{rank}(A)$. Therefore, $\text{rank}(A+B) \ge |\text{rank}(A) - \text{rank}(B)|$. □

Theorem 3.3.7

Theorem 3.3.7. Let A be an $n \times n$ full rank matrix. Then $(A^{-1})^T = (A^T)^{-1}$.

Proof. First, A^T is also $n \times n$ and full rank by Theorem 3.3.2. We have

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T \text{ by Theorem 3.2.1(1)} \\ &= \mathcal{I}^T = \mathcal{I}, \end{aligned}$$

so a right inverse of A^T is $(A^{-1})^T$. Since A is full rank and square then, as discussed above, $(A^T)^{-1} = (A^{-1})^T$. □

Theorem 3.3.8

Theorem 3.3.8. $n \times m$ matrix A, where $n \le m$, has a right inverse if and only if A is of full row rank n. $n \times m$ matrix A, where $m \le n$, has a left inverse if and only if A has full column rank m.

Proof. Let A be an $n \times m$ matrix where $n \le m$ and let A be of full row rank (that is, $\text{rank}(A) = n$). Then the column space of A, $\mathcal{V}(A)$, is of dimension n and each e_i, where e_i is the ith unit vector in $\mathbb{R}^n$, is in $\mathcal{V}(A)$ so that there is $x_i \in \mathbb{R}^m$ such that $Ax_i = e_i$ for $i = 1, 2, \ldots, n$. With X an $m \times n$ matrix with columns x_i and the columns of I_n as e_i, we have $AX = I_n$. Also, by Theorem 3.3.6, $n = \text{rank}(I_n) \le \min\{\text{rank}(A), \text{rank}(X)\}$ where $\text{rank}(A) = n$, so $\text{rank}(X) = n$ and X is of full column rank. Furthermore, $AX = I_n$ has a solution only if A has full row rank n since the n columns of I_n are linearly independent. That is, A has a right inverse if and only if A is of full row rank. The result similarly follows for the left inverse claim. □

Theorem 3.3.9

Theorem 3.3.9. If A is an $n \times m$ matrix of rank $r > 0$ then there are matrices P and Q, both products of elementary matrices, such that PAQ is the equivalent canonical form of A, $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Proof. We prove this by induction. Since $\text{rank}(A) > 0$ then some $a_{ij} \ne 0$. We move this into position $(1,1)$ by interchanging row 1 and i and interchanging columns 1 and j to produce $E_{1i}AE_{1j}^c$ (we use superscripts of 'c' to denote column operations). Then divide the first row by a_{ij} to produce an entry of 1 in the $(1,1)$ position (we denote the corresponding elementary matrix as $E_{(1/a_{ij})1}$) to produce $B = E_{(1/a_{ij})1}E_{1i}AE_{1j}^c$. Next we "eliminate" the entries in the first column of B under the $(1,1)$ entry with the elementary row operations $R_k \to R_k - b_{k1}R_1$ for $2 \le k \le n$ (we denote the corresponding elementary row matrices as $E_{k(-b_{n1})1}$ for $2 \le k \le n$) to produce

$$C = E_{n(-b_{n1})1}E_{(n-1)(-b_{(n-1)1})1} \cdots E_{2(-b_{21})1}B.$$

Theorem 3.3.9 (continued 1)

Proof (continued). Similarly we eliminate the entries in the first row of C to the right of the $(1,1)$ entry with the elementary column operations $C_k \to C_k - c_{1k}C_1$ (with the corresponding elementary matrices $E_{n(-c_{1n})1}^c$) to produce

$$CE_{2(-c_{12})1}^c E_{3(-c_{13})1}^c \cdots E_{n(-c_{1n})1}^c.$$

We now have a matrix of the form $P_1AQ_1 = \begin{bmatrix} I_1 & 0_{R_1} \\ 0_{C_1} & X_1 \end{bmatrix}$ where 0_{R_1} is $1 \times (n-1)$, 0_{C_1} is $(n-1) \times 1$, and X is $(n-1) \times (n-1)$. Also, P_1 and Q_1 are products of elementary matrices. By Theorem 3.3.3, $\text{rank}(A) = \text{rank}(P_1AQ_1) = r$. Since $\mathcal{V}\left(\begin{bmatrix} I_1 \\ 0_{C_1} \end{bmatrix}\right) \perp \mathcal{V}\left(\begin{bmatrix} 0_{R_1} \\ X_1 \end{bmatrix}\right)$ then by Theorem 3.3.4(iv) $r = \text{rank}\left(\begin{bmatrix} I_1 \\ 0_{C_1} \end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix} 0_{R_1} \\ X_1 \end{bmatrix}\right) = 1 + \text{rank}\left(\begin{bmatrix} 0_{R_1} \\ X_1 \end{bmatrix}\right)$ and so $\text{rank}\left(\begin{bmatrix} 0_{R_1} \\ X_1 \end{bmatrix}\right) = r - 1$.

Theorem 3.3.9 (continued 2)

Proof (continued). So $\text{rank}(X_1) = r - 1$ (also by Theorem 3.3.4(iv), if you like). If $r - 1 > 0$ then we can similarly find P_2 and Q_2 products of elementary matrices such that

$$P_2 P_1 A Q_1 Q_2 = \begin{bmatrix} I_2 & 0_{R_2} \\ 0_{C_2} & X_2 \end{bmatrix}$$

and $\text{rank}(X_2) = r - 2$. Continuing this process we can produce

$$P_r P_{r-1} \cdots P_1 A Q_1 Q_2 \cdots Q_r = \begin{bmatrix} I_r & 0_{R_r} \\ 0_{C_r} & X_r \end{bmatrix}$$

where X_r has rank 0; that is, where X_r is a matrix of all 0's. So

$$P_r P_{r-1} \cdots P_1 A Q_1 Q_2 \cdots Q_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

as claimed. □

Theorem 3.3.11

Theorem 3.3.11. If A is a square full rank matrix (that is, nonsingular) and if B and C are conformable matrices for the multiplications AB and CA then $\text{rank}(AB) = \text{rank}(B)$ and $\text{rank}(CA) = \text{rank}(C)$.

Proof. By Theorem 3.3.5, $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \leq \text{rank}(B)$. Also, $B = A^{-1}AB$ so by Theorem 3.3.5, $\text{rank}(B) \leq \min\{\text{rank}(A^{-1}), \text{rank}(AB)\} \leq \text{rank}(AB)$. So $\text{rank}(B) = \text{rank}(AB)$.

Similarly, $\text{rank}(CA) \leq \text{rank}(C)$ and $C = CAA^{-1}$ so $\text{rank}(C) \leq \text{rank}(CA)$ and hence $\text{rank}(C) = \text{rank}(CA)$. □

Theorem 3.3.12

Theorem 3.3.12. If A is a full column rank matrix and B is conformable for the multiplication AB, then $\text{rank}(AB) = \text{rank}(B)$. If A is a full row rank matrix and C is conformable for the multiplication CA, then $\text{rank}(CA) = \text{rank}(C)$.

Proof. Let A be $n \times m$ and of full column rank $m \leq n$. By Theorem 3.3.8, A has a left inverse A_L^{-1} where $A_L^{-1}A = I_m$. By Theorem 3.3.5, $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \leq \text{rank}(B)$. Now $B = I_m B = A_L^{-1}AB$, so by Theorem 3.3.5 $\text{rank}(B) \leq \min\{\text{rank}(A_L^{-1}), \text{rank}(AB)\} \leq \text{rank}(AB)$, and so $\text{rank}(AB) = \text{rank}(B)$.

Next let A be $n \times m$ and of row column rank $n \leq m$. By Theorem 3.3.8, A has a right inverse A_R^{-1} where $AA_R^{-1} = I_n$. By Theorem 3.3.5, $\text{rank}(CA) \leq \text{rank}(C)$. Now $C = CI_n = CAA_R^{-1}$, so by Theorem 3.3.5 $\text{rank}(C) \leq \text{rank}(CA)$ and so $\text{rank}(CA) = \text{rank}(C)$. □

Theorem 3.3.13

Theorem 3.3.13. Let C be $n \times n$ and positive definite and let A be $n \times m$.

(1) If C is positive definite and A is of full column rank $m \leq n$ then $A^T CA$ is positive definite.

(2) If $A^T CA$ is positive definite then A is of full column rank $m \leq n$.

Proof. (1) Let $x \in \mathbb{R}^m$, where $x \neq 0$, and let $y = Ax$. So y is a linear combination of the columns of A and since A is of full column rank (so that the columns of A form a basis for the column space of A) and $x \neq 0$ implies $y \neq 0$. Since C is hypothesized to be positive definite,

$$x^T(A^T CA)x = (Ax)^T C(Ax) = y^T Cy > 0.$$

Also, $A^T CA$ is $m \times m$ and symmetric since $(A^T CA)^T = A^T C^T (A^T)^T = A^T CA$. Therefore $A^T CA$ is positive definite.

Theorem 3.3.13 (continued)

Theorem 3.3.13. Let C be $n \times n$ and positive definite and let A be $n \times m$.

(1) If C is positive definite and A is of full column rank $m \leq n$ then $A^T CA$ is positive definite.

(2) If $A^T CA$ is positive definite then A is of full column rank $m \leq n$.

Proof (continued). (2) ASSUME not; assume that A is not of full column rank. Then the columns of A are not linearly independent and so with $a_1, a_2, \ldots, a_m$ as the columns of A, there are scalars $x_1, x_2, \ldots, x_m$ not all 0, such that $x_1 a_1 + x_2 a_2 + \cdots + x_m a_m = 0$. But then $x \in \mathbb{R}^m$ with entries x_i satisfies $x \neq 0$ and $Ax = 0$. Therefore $x^T(A^T CA)x = (x^T A^T C)(Ax) = (x^T A^T C)0 = 0$, and so $A^T CA$ is not positive definite, a CONTRADICTION. So the assumption that A is not of full column rank is false. Hence, A is of full column rank. □

Theorem 3.3.14

Theorem 3.3.14. Properties of $A^T A$.
Let A be an $n \times m$ matrix.

(1) $A^T A = 0$ if and only if $A = 0$.
(2) $A^T A$ is nonnegative definite.
(3) $A^T A$ is positive definite if and only if A is of full column rank.
(4) $(A^T A)B = (A^T A)C$ if and only if $AB = AC$, and $B(A^T A) = C(A^T A)$ if and only if $BA^T = CA^T$.
(5) $A^T A$ is of full rank if and only if A is of full column rank.
(6) $\text{rank}(A^T A) = \text{rank}(A)$.

The product $A^T A$ is called a *Gramian matrix*.

Proof. (1) If $A = 0$ then $A^T = 0$ and $A^T A = 00 = 0$. If $A^T A = 0$ then $\text{tr}(A^T A) = 0$. Now the (i,j) entry of $A^T A$ is $\sum_{k=1}^n a^t_{ik} a_{kj} = \sum_{k=1}^n a_{ki} a_{kj}$ and so the diagonal (i,i) entry is $\sum_{k=1}^n a_{ki}^2$. Then

$$0 = \text{tr}(A^T A) = \sum_{i=1}^m \sum_{k=1}^n a_{ki}^2 = \sum_{i=1}^m \sum_{j=1}^n a_{ji}^2 = \sum_{j=1}^m \sum_{i=1}^n a_{ij}^2 \ldots$$

Theorem 3.3.14 (continued 1)

Proof (continued). ...and so $a_{ij} = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$; that is, $A = 0$.

(2) For any $y \in \mathbb{R}^m$ we have

$$y^T(A^T A)y = (Ay)^T(Ay) = \|Ay\|^2 \geq 0.$$

(3) From (2), $y^T(A^T A)y = \|Ay\|^2$, so $y^T(A^T A)y = 0$ if and only if $\|Ay\| = 0$. Now Ay is a linear combination of the columns of A so if A is of full column rank then $Ay = 0$ if and only if $y = 0$. That is, if A is of full column rank then for $y \neq 0$ we have $y^T(A^T A)y = \|Ay\|^2 > 0$ and $A^T A$ is positive definite.
If A is not of full column rank then the columns of A are not linearly independent and with $a_1, a_2, \ldots, a_n$ as the columns of A, there are scalars $y_1, y_2, \ldots, y_n$, not all 0, such that $y_1 a_1 + y_2 a_2 + \cdots + y_n a_n = 0$. Then the $y \in \mathbb{R}^n$ with entries y_i we have $y \neq 0$ and $Ay = 0$. Then $y^T(A^T A)y = \|Ay\|^2 = 0$, and so $A^T A$ is not positive definite.

Theorem 3.3.14 (continued 2)

Proof (continued). (4) Suppose $A^T AB = A^T AC$. Then

$A^T AB - A^T AC = 0$ or $A^T A(B - C) = 0$, and so $(B^T - C^T)A^T A(B - C) = 0$. Hence $(A(B - C))^T(A(B - C)) = 0$ and by Part (1), $A(B - C) = 0$. That is, $AB = AC$. Conversely, if $AB = AC$ then $A^T AB = A^T AC$. Therefore $A^T AB = A^T AC$ if and only if $AB = AC$. Now suppose $BA^T A = CA^T A$. Then $BA^T A - CA^T A = 0$ or $(B - C)A^T A = 0$, and so $(B - C)A^T A(B^T - C^T) = 0$. Hence $((B - C)A^T)((B - C)A^T)^T = 0$ and by Part (1), $(B - C)A^T = 0$. That is, $BA^T = CA^T$. Conversely, if $BA^T = CA^T$ then $BA^T A = CA^T A$. Therefore $BA^T A = CA^T A$ if and only if $BA^T = CA^T$.

(5) Suppose A is of full column rank $m \leq n$. Then by Theorem 3.3.12, $\text{rank}(A^T A) = \text{rank}(A) = m$. Since $A^T A$ is $m \times m$, then $A^T A$ is of full rank.

Theorem 3.3.14 (continued 3)

Proof (continued). Now suppose A^TA if of full rank m. Then by Theorem 3.3.5, $m = \text{rank}(A^TA) \le \min\{\text{rank}(A^T), \text{rank}(A)\} \le \text{rank}(A)$, and since A is $n \times m$ then A must be of full column rank m.

(6) Let $\text{rank}(A) = r$. If $r = 0$ then $A = 0$ and so $A^TA = 0$ and $\text{rank}(A^TA) = 0$ and the claim holds. If $r > 0$, then the columns of A can be permuted so that the first r columns are linearly independent. That is, there is a permutation matrix Q such that $AQ = [A_1\, A_2]$ where A_1 is an $n \times r$ matrix of rank r (and by Theorem 3.3.3, $\text{rank}(AQ) = \text{rank}(A) = r$). So A_1 is of full column rank and so each column of A_2 is in the column space of A_1. So there is $r \times (m-r)$ matrix B such that $A_2 = A_1B$. Then $AQ = [A_1\, A_2] = [A_1I_r\, A_1B] = A_1[I_r\, B]$. Hence

$(AQ)^T = (A_1[I_r\, B])^T = \begin{bmatrix} I_r \\ B^T \end{bmatrix} A_1^T$ and

$(AQ)^T(AQ) = \begin{bmatrix} I_r \\ B^T \end{bmatrix} A_1^TA_1[I_r\, B]$. Define $T = \begin{bmatrix} I_r & 0 \\ -B^T & I_{m-r} \end{bmatrix}$.

Theorem 3.3.14 (continued 4)

Proof (continued). Then T is $m \times m$ and of full rank m (as is T^T), so by Theorem 3.3.12

$$\text{rank}(A^TA) = \text{rank}((AQ)^T(AQ))$$

$$= \text{rank}(T(AQ)^T(AQ)) = \text{rank}(T(AQ)^T(AQ)T^T). \qquad (*)$$

Now

$$T(AQ)^T = \begin{bmatrix} I_r & 0 \\ -B^T & I_{m-r} \end{bmatrix} \begin{bmatrix} I_r \\ B^T \end{bmatrix} A_1^T = \begin{bmatrix} I_rI_r + 0B^T \\ -B^TI_r + I_{m-r}B^T \end{bmatrix} A_1^T$$

$$= \begin{bmatrix} I_r \\ 0 \end{bmatrix} A_1^T = \begin{bmatrix} A_1^T \\ 0 \end{bmatrix}$$

and

$$(AQ)T^T = (T(AQ)^T)^T = \begin{bmatrix} A_1^T \\ 0 \end{bmatrix}^T = [A_1\, 0].$$

Theorem 3.3.14 (continued 5)

Proof (continued). So

$$T(AQ)^T(AQ)T^T = \begin{bmatrix} A_1^T \\ 0 \end{bmatrix} [A_1\, 0] = \begin{bmatrix} A_1^TA_1 & 0 \\ 0 & 0 \end{bmatrix}$$

(the matrix products are justified by Theorem 3.2.2). So by $(*)$,

$$\text{rank}(A^TA) = \text{rank}\left(\begin{bmatrix} A_1^TA_1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{rank}(A_1^TA_1).$$

Since A_1 is of full column rank r, by Part (5) $A_1^TA_1$ is of full rank r. So $\text{rank}(A^TA) = \text{rank}(A_1^TA_1) = r = \text{rank}(A)$, as claimed. □

Theorem 3.3.15

Theorem 3.3.15. If A is a $n \times n$ matrix and B is $n \times \ell$ then $\text{rank}(AB) \ge \text{rank}(A) + \text{rank}(B) - n$.

Proof. Let $r = \text{rank}(A)$. By Theorem 3.3.9, there are $n \times n$ matrices P and Q which are products of elementary matrices such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Let $C = P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^{-1}$ and then

$$A+C = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} + P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^{-1} = P^{-1}I_nQ^{-1} = P^{-1}Q^{-1}.$$

Now P^{-1} and Q^{-1} are of full rank n (see the notes before the definition of inverse matrix), so by Theorem 3.3.11,

$$\text{rank}(C) = \text{rank}\left(\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}\right) = \text{rank}(I_{n-r}) = n - \text{rank}(A). \qquad (*)$$

Theorem 3.3.15 (continued)

Theorem 3.3.15. If A is a $n \times n$ matrix and B is $n \times \ell$ then $\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n$.

Proof (continued). So for $n \times \ell$ matrix B,

$$\begin{aligned} \text{rank}(B) &= \text{rank}(P^{-1}Q^{-1}B) \text{ by Theorem 3.3.11} \\ &= \text{rank}(AB + CB) \text{ since } A + C = P^{-1}Q^{-1} \\ &\leq \text{rank}(AB) + \text{rank}(CB) \text{ by Theorem 3.3.6} \\ &\leq \text{rank}(AB) + \text{rank}(C) \text{ by Theorem 3.3.5} \\ &= \text{rank}(AB) + n - \text{rank}(A) \text{ by } (*). \end{aligned}$$

So $\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB)$. □

Theorem 3.3.16

Theorem 3.3.16. $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Proof. By Theorem 3.2.4, $\det(AB) = \det(A)\det(B)$, so if A^{-1} exists then $\det(A) = 1/\det(A^{-1})$ and so $\det(A) \neq 0$.

Conversely, if $\det(A) \neq 0$ then by Theorem 3.1.3, $A^{-1} = (1/\det(A))\text{adj}(A)$ and A is invertible. □

Theorem 3.3.18

Theorem 3.3.18. If A and B are $n \times n$ full rank matrices then the Kronecker product satisfies $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Proof. Since A and B are full rank, then A^{-1} and B^{-1} exist. Let $A = [a_{ij}]$ and $A^{-1} = [c_{ij}]$. Then $(A \otimes B)(A^{-1} \otimes B^{-1})$

$$\begin{aligned} &= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} \begin{bmatrix} c_{11}B^{-1} & c_{12}B^{-1} & \cdots & c_{1n}B^{-1} \\ c_{21}B^{-1} & c_{22}B^{-1} & \cdots & c_{2n}B^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}B^{-1} & c_{n2}B^{-1} & \cdots & c_{nn}B^{-1} \end{bmatrix} \\ &= \left[\sum_{k=1}^{n} a_{ik}c_{kj}I_n \right] \text{ since } (a_{ik}B)(c_{kj}B^{-1}) = a_{ik}c_{kj}I_n \\ &= I_{n^2}, \end{aligned}$$

and so $A^{-1} \otimes B^{-1} = (A \otimes B)^{-1}$. □