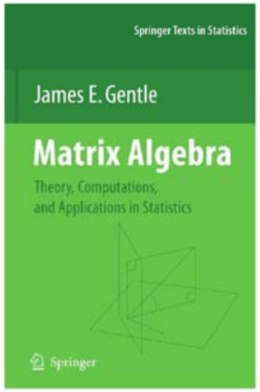
Theory of Matrices

Chapter 3. Basic Properties of Matrices
3.4. More on Partitioned Square Matrices: The Schur Complement—Proofs of Theorems



Theorem 3.4.2

Theorem 3.4.2. If A is a square matrix such that $A = \begin{bmatrix} X^T \\ y^T \end{bmatrix} [X\ y]$ where X is of full column rank, then the Schur complement of X^TX in A is

$$y^Ty - y^TX(X^TX)^{-1}X^Ty.$$

Proof. By Theorem 3.2.2 we have

$$A = \begin{bmatrix} X^T \\ y^T \end{bmatrix} [X\ y] = \begin{bmatrix} X^TX & X^Ty \\ y^TX & y^Ty \end{bmatrix},$$

so the Schur complement of X^TX in A is, by definition, $Z = y^Ty - y^TX(X^TX)^{-1}X^Ty$, as claimed. □

Theorem 3.4.3

Theorem 3.4.3. If A is a square matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11} is square and nonsingular then

$$\det(A) = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = \det(A_{11})\det(Z)$$

where $Z = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of A_{11} in A.

Proof. By Theorem 3.2.2, we can write A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}.$$

So by Theorem 3.2.4,

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}\right) \det\left(\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}\right) \\ &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12})\det(I)\det(I) \text{ by Theorem 3.1.G} \\ &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) \text{ since } \det(I) = 1. \end{aligned}$$

□