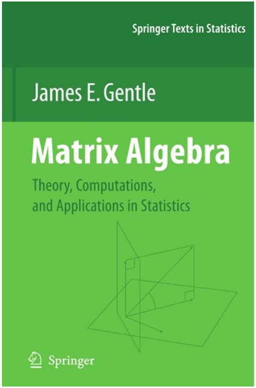
Theory of Matrices

Chapter 3. Basic Properties of Matrices

3.4. More on Partitioned Square Matrices: The Schur Complement—Proofs of Theorems



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$$y^T y - y^T X (X^T X)^{-1} X^T y.$$

Proof. By Theorem 3.2.2 we have

$$A = \begin{bmatrix} X^T \\ y^T \end{bmatrix} [X\ y] = \begin{bmatrix} X^T X & X^T y \\ y^T X & y^T y \end{bmatrix},$$

so the Schur complement of $X^T X$ in A is, by definition, $Z = y^T y - y^T X (X^T X)^{-1} X^T y$, as claimed. □

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$$\det(A) = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = \det(A_{11})\det(Z)$$

where $Z = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of A_{11} in A.

Proof. By Theorem 3.2.2, we can write A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}.$$

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So by Theorem 3.2.4,

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}\right)\det\left(\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}\right) \\ &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12})\det(I)\det(I) \text{ by Theorem 3.1.G} \\ &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) \text{ since } \det(I) = 1. \quad \square \end{aligned}$$

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So by Theorem 3.2.4,

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}\right) \det\left(\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}\right) \\ &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12})\det(I)\det(I) \text{ by Theorem 3.1.G} \\ &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) \text{ since } \det(I) = 1. \quad \square \end{aligned}$$