Theorem 3.5.1

**Theorem 3.5.1.** If $Ax = b$ is an underdetermined system then there are an infinite number of solutions to the system.

**Proof.** If $Ax = b$ is an underdetermined system with $A$ $n \times m$ then, since it is consistent by definition, there is a solution $x_1$ such that $Ax_1 = b$. Since $\text{rank}(A) < m$ and $A$ has $m$ columns, then by Exercise 2.1 the columns of $A$ are not linearly independent. So with $a_1, a_2, \ldots, a_m$ as the columns of $A$, there are scalars $s_1, s_2, \ldots, s_m$ not all 0 for which $s_1a_1 + s_2a_2 + \cdots + s_ma_m = 0$. Let $s \in \mathbb{R}^m$ have components $s_i$, and define $x_2 = s + x_1$. Then $Ax_2 = A(s + x_1) = As + Ax_1 = 0 + b = b$ and so $x_2$ is also a solution to the system $Ax = b$. Now let $w \in \mathbb{R}$ and consider $x_w = wx_1 + (1 - w)x_2$. We have

$$Ax_w = A(wx_1 + (1 - w)x_2) = wAx_1 + (1 - w)Ax_2 = wb + (1 - w)b = b$$

and each $x_w$ is a solution to $Ax = b$. Therefore, $Ax = b$ has an infinite number of solutions. □

Theorem 3.5.2

**Theorem 3.5.2.** Properties of the Generalized Inverse.

(1) If $A^-$ is a generalized inverse of $A$ then $(A^-)^T$ is a generalized inverse of $A^T$.

(2) $(A^-A)(A^-A) = A^-A$; that is, $A^-A$ is idempotent.

(3) $\text{rank}(A^-A) = \text{rank}(A)$.


(5) $\text{rank}(I - A^-A) = m - \text{rank}(A)$ where $A$ is $n \times m$.

**Proof.** (1) We have $A = AA^-A$ so, by Theorem 3.2.1(1),

$$A^T = (AA^-A)^T = A^T(A^-)^T A^T$$

and so $(A^-)^T$ is a generalized inverse of $A^T$.


(3) By Theorem 3.3.5, $\text{rank}(A^-A) \leq \min\{\text{rank}(A^-), \text{rank}(A)\} \leq \text{rank}(A)$.

Since $A = AA^-A$ then again by Theorem 3.3.5,

$$\text{rank}(A) \leq \min\{\text{rank}(A), \text{rank}(A^-A)\} \leq \text{rank}(A^-A)$$

and so $\text{rank}(A) = \text{rank}(A^-A)$.

(4) We have $(I - A^-A)(A^-A) =


So


(5) Notice that $A^-A$ is $m \times m$ and by Part (4) $(I - A^-A)A^-A = 0$, so

$$0 = \text{rank}(0) = \text{rank}((I - A^-A)A^-A)$$

$$\geq \text{rank}(I - A^-A) + \text{rank}(A^-A) - m \text{ by Theorem 3.3.15}$$

$$= \text{rank}(I - A^-A) + \text{rank}(A) - m \text{ by Part (3)} \quad (\ast)$$

Next, $I = I - A^-A + A^-A$ and by Theorem 3.3.6,

$$m = \text{rank}(I) = \text{rank}(I - A^-A + A^-A) \leq \text{rank}(I - A^-A) + \text{rank}(A^-A)$$

$$= \text{rank}(I - A^-A) + \text{rank}(A) \text{ by Part (3)}. \quad (\ast\ast)$$

Combining (\ast) and (\ast\ast) gives $m = \text{rank}(I - A^-A) + \text{rank}(A)$ and the claim follows. □
Theorem 3.5.3

Theorem 3.5.3. Let $Ax = b$ be a consistent system of equations and let $A^{-}$ be a generalized inverse of $A$.

1. $x - A^{-}b$ is a solution.
2. If $x = Gb$ is a solution of system $Ax = b$ for all $b$ such that a solution exists, then $AGA = A$; that is, $G$ is a generalized inverse of $A$.
3. For any $z \in \mathbb{R}^m$, $A^{-}b + (I - A^{-}A)z$ is a solution.
4. Every solution is of the form $x = A^{-}b + (I - A^{-}A)z$ for some $z \in \mathbb{R}^m$.

Proof. (1) We have $(AA^{-})x = Ax$ and with $Ax = b$ as the given system, we get $AA^{-}(Ax) = Ax$ or $AA^{-}b = b$ or $A(A^{-}b) = b$; that is, $A^{-}b$ is a solution to $Ax = b$.

Theorem 3.5.4

Theorem 3.5.4. The nullity of $n \times m$ matrix $A$ satisfies $\dim(\mathcal{N}(A)) = m - \operatorname{rank}(A)$.

Proof. If $x \in \mathcal{N}(A)$ then $Ax = 0$ and by Theorem 3.5.3 (3 and 4) $x = 0 + (I - A^{-}A)z = (I - A^{-}A)z$ for any $z \in \mathbb{R}^m$ (and conversely every solution to $Ax = 0$ is of this form). Now $(I - A^{-}A)z$ is in the column space of $I - A^{-}A$ for every $z \in \mathbb{R}^m$, so by Theorem 3.5.2(5),

$$\dim(\mathcal{N}(A)) = \operatorname{rank}(I - A^{-}A) = m - \operatorname{rank}(A).$$

Theorem 3.5.5

Theorem 3.5.5. (1) If system $Ax = b$ is consistent, then any solution is of the form $x = A^{-}b + z$ for some $z \in \mathcal{N}(A)$.

(2) For matrix $A$, the null space of $A$ is orthogonal to the row space of $A$: $\mathcal{N}(A) \perp \mathcal{V}(A^T)$.

(3) For matrix $A$, $\mathcal{N}(A) \oplus \mathcal{V}(A^T) = \mathbb{R}^m$.

Proof. (1) Let $y$ be a solution of $Ax = b$. Then $Ay = b = AA^{-}b$ by Theorem 3.5.3(1) and so $Ay - AA^{-}b = A(y - A^{-}b) = 0$. Therefore $z = y - A^{-}b \in \mathcal{N}(A)$. So $y = A^{-}b + z$ where $z \in \mathcal{N}(A)$.

(2) Let $a \in \mathcal{V}(A^T)$ and $b \in \mathcal{N}(A)$. Then

$$\langle b, a \rangle = b^T a = b^T A^T s$$

since $a \in \mathcal{V}(A^T)$ then $a = A^T s$ for some $s \in \mathbb{R}^n$

$$= (b^T A^T) s = (Ab)^T s$$

by Theorem 3.2.1(1)

$$= 0s = 0$$

since $b \in \mathcal{N}(A)$. 

Proof (continued). (2) Let the columns of $A$ be $a_1, a_2, \ldots, a_m$. The $m$ systems $Ax = a_j$ (where $1 \leq j \leq n$) each have a solution (namely, the $j$th unit vector in $\mathbb{R}^m$). So by hypothesis, $Ga_j$ is a solution of the system $Ax = a_j$ for each $j$ (where $1 \leq j \leq n$). That is, $AGa_j = a_j$ for $1 \leq j \leq n$, or $AGA = A$.

(3) We have

$$A(A^{-}b + (I - A^{-}A)z) = AA^{-}b + (A - AA^{-}A)z$$

$$= b + (A - A)z$$

by Part (1)

$$= b + 0 = b.$$

(4) Let $y$ be a solution of $Ax - b$. Then

$$y = A^{-}b - A^{-}b + y = A^{-}b - A^{-}(Ay) + y$$

since $Ay = b$

$$= A^{-}b - (A^{-}A - I)y = A^{-}b + (I - A^{-}A)z$$

with $z = y$. 

\[ \square \]
Theorem 3.5.5.

(1) If system $Ax = b$ is consistent, then any solution is of the form $x = A^{-1}b + z$ for some $z \in \mathcal{N}(A)$.

(2) For matrix $A$, the null space of $A$ is orthogonal to the row space of $A$: $\mathcal{N}(A) \perp \mathcal{V}(A^T)$.

(3) For matrix $A$, $\mathcal{N}(A) \oplus \mathcal{V}(A^T) = \mathbb{R}^m$.

Proof (continued). So $a \perp b$. Since $a$ is an arbitrary element of $\mathcal{V}(A^T)$ and $b$ is an arbitrary element of $\mathcal{N}(A)$ then $\mathcal{N}(A) \perp \mathcal{V}(A^T)$.

(3) From Theorem 3.5.4 (the rank-nullity equation), $\dim(\mathcal{N}(A)) + \dim(\mathcal{V}(A^T)) = m$. Now both $\mathcal{N}(A)$ and $\mathcal{V}(A^T)$ are subspaces of $\mathbb{R}^m$, so $\mathcal{N}(A) \oplus \mathcal{V}(A^T)$ is a $m$ dimensional subspace of $\mathbb{R}^m$. That is, $\mathcal{N}(A) \oplus \mathcal{V}(A^T) = \mathbb{R}^m$ (technically, we need the Fundamental Theorem of Finite Dimensional Vector Spaces here). \qed