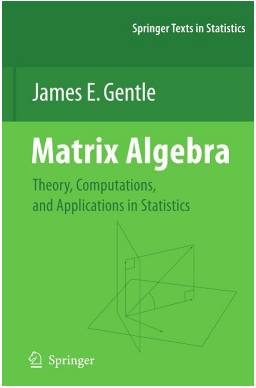
# Theory of Matrices

**Chapter 3. Basic Properties of Matrices**

3.6. Generalized Inverses—Proofs of Theorems



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# Theorem 3.6.1 (continued)

**Theorem 3.6.1.** Let $A$ be an $n \times m$ matrix. Then a generalized inverse of $A$ exists.

**Proof (continued).** Then

$$A\left(Q^{-1}\begin{bmatrix} I_r & U \\ V & W \end{bmatrix}P^{-1}\right)A$$

$$= P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}QQ^{-1}\begin{bmatrix} I_r & U \\ V & W \end{bmatrix}P^{-1}P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q$$

$$= P\begin{bmatrix} I_r & U \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q$$

$$= P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q = A$$

and so $A^{-} = Q^{-1}\begin{bmatrix} I_r & U \\ V & W \end{bmatrix}P^{-1}$ is a generalized inverse of $A$. □

# Theorem 3.6.2

**Theorem 3.6.2.** Every matrix $A$ with $\text{rank}(A) = r > 0$ has a pseudoinverse given be $A^+ = R^T(L^TAR^T)^{-1}L^T$ where $A = LR$ is a full rank factorization of $A$ (such a factorization exists as shown in equations $(**)$ and $(***)$ of Section 3.4).

**Proof.** Let $A$ be $n \times m$ with $\text{rank}(A) = r > 0$. Then there is a full rank factorization of $A$, $A = LR$ where $L$ is a $n \times r$ full column rank matrix and $R$ is a $r \times m$ full row rank matrix.

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# Theorem 3.6.3

**Theorem 3.6.3.** For any matrix $A$, the pseudoinverse $A^+$ is unique.

**Proof.** The case $A = 0$ is addressed in the note above. For $A$ with $\text{rank}(A) > 0$, suppose both $B$ and $C$ are pseudoinverses. We use the four properties of a pseudoinverse to prove that $B = C$.

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$$\begin{aligned} B &= BAB \text{ by Property (2) for } B \\ &= (BA)^T B = A^T B^T B \text{ by Property (3) for } B \\ &= (ACA)^T B^T B \text{ by Property (1) for } C \\ &= A^T C^T A^T B^T B = (CA)^T A^T B^T B \\ &= CAA^T B^T B \text{ by Property (3) for } C \\ &= CA(BA)^T B = CA(BA)B \text{ by Property (3) for } B \\ &= C(ABA)B = CAB \text{ by Property (1) for } B \end{aligned}$$

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# Theorem 3.6.3 (continued)

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$$\begin{aligned} CAB &= C(ACA)B = C(AC)AB \text{ by Property (1) for } C \\ &= C(AC)^T AB = CC^T A^T AB \text{ by Property (3) for } C \\ &= CC^T A^T (AB)^T = CC^T (ABA)^T \text{ by Property (4) for } B \\ &= CC^T A^T = C(AC)^T \text{ by Property (1) for } B \\ &= CAC \text{ by Property (4) for } C \\ &= C \text{ by Property (2) for } C. \end{aligned}$$

So $B = C$ and the pseudoinverse of $A$ is unique. □