Theorem 3.7.1

Theorem 3.7.1. Let $Q$ be an $n \times m$ matrix. For $n \leq m$, $Q$ is orthogonal if and only if $QQ^T = I_n$ for $n \geq m$, $Q$ is orthogonal if and only if $Q^TQ = I_m$. A square matrix $Q$ is orthogonal if and only if $QQ^T = Q^TQ = I$ (so a square matrix $Q$ is orthogonal if and only if it is invertible and $Q^{-1} = Q^T$).

Proof. First, suppose $n \leq m$. If $Q$ is orthogonal then the row rank of $Q$ equals the column rank of $Q$ by Theorem 3.3.2 and so $Q$ must have $n$ orthonormal rows (since it cannot have more orthonormal [and hence linearly independent] columns than rows). Let the rows of $Q$ be $r_1, r_2, \ldots, r_n$. Then the columns of $Q^T$ are $r_1^T, r_2^T, \ldots, r_n^T$. So the $(i,j)$ entry of $QQ^T$ is $r_i^Tr_j = \langle r_i, r_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ since the rows of $Q$ are orthonormal. Since $QQ^T$ is $n \times n$, then $QQ^T = I_n$. Conversely, if $QQ^T = I_n$, then the $(i,j)$ entry of $QQ^T$ is $\langle r_i, r_j \rangle$ as given above and so the rows of $Q$ form an orthonormal set and $Q$ is orthogonal.

Theorem 3.7.1 (continued)

Proof (continued). Second, suppose $n \geq m$. If $Q$ is orthogonal then, similar to the case $n \leq m$, it must be that $Q$ has $m$ orthonormal columns. Let the columns of $Q$ be $c_1, c_2, \ldots, c_m$. Then the rows of $Q^T$ are $c_1^T, c_2^T, \ldots, c_m^T$. So the $(i,j)$ entry of $Q^TQ$ is $c_i^Tc_j = \langle c_i, c_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ since the columns of $Q$ are orthonormal.

Since $Q^TQ$ is $m \times m$ then $Q^TQ = I_m$. Conversely, if $Q^TQ = I_m$ then the $(i,j)$ entry of $Q^TQ$ is $\langle c_i, c_j \rangle$ as given above and so the columns of $Q$ form an orthonormal set and $Q$ is orthogonal.

If $Q$ is $n \times n$, then combining the first two cases we have that $Q$ is orthogonal if and only if $QQ^T = I = Q^TQ$.

Corollary 3.7.2

Corollary 3.7.2. For $Q$ a square orthogonal matrix, we have $\det(Q) = \pm 1$. For $Q$ an $n \times m$ orthogonal matrix $Q$ with $n \geq m$, we have $\langle Q, Q \rangle = m$.

Proof. by Theorem 3.7.1 and Theorem 3.2.4, $\det(QQ^T) = \det(I)$ or $\det(Q)^2 = 1$. By Theorem 3.1.4, $\det(Q^T) = \det(Q)$, so $\det(Q)^2 = 1$ and $\det(Q) = \pm 1$.

Let the columns of $n \times m$ orthogonal $Q$ be $c_1, c_2, \ldots, c_m$. Then

$$\langle Q, Q \rangle = \sum_{j=1}^{n} c_j^Tc_j = \sum_{j=1}^{m} \langle c_j, c_j \rangle = \sum_{j=1}^{m} \|c_j\|^2 = m$$

since the columns of $Q$ form an orthonormal set of vectors.
Theorem 3.7.3

Theorem 3.7.3. Every permutation matrix is orthogonal.

Proof. First, form the elementary permutation matrix $E_{pq}$ from $I_n$ by interchanging rows $p$ and $q$ of $I_n$, $I_n \xrightarrow{R_p \leftrightarrow R_q} E_{pq} = [e_{ij}]$. So we have $e_{ij} = 0$ for $i \in \{1, 2, \ldots, n\} \setminus \{p, q\}$ and $i \neq j$, and $e_{ij} = 1$ for $i \in \{1, 2, \ldots, n\} \setminus \{p, q\}$. For $i = p$ we have $e_{pj} = 0$ for $j \neq q$ and $e_{pq} = 1$. For $i = q$ we have $e_{qj} = 0$ for $j \neq p$ and $e_{qp} = 1$. So in $E_{pq}^T = [e_{ji}^T]$, we have $e_{ij}^T = 0$ for $i \in \{1, 2, \ldots, n\} \setminus \{p, q\}$ and $i \neq j$, and $e_{ij}^T = 1$ for $i \in \{1, 2, \ldots, n\} \setminus \{p, q\}$. For $i = p$ we have $e_{ip}^T = e_{pj} = 0$ for $j \neq q$ and $e_{pq} = 1$. For $i = q$ we have $e_{iq}^T = e_{qj} = 0$ for $j \neq p$ and $e_{qp} = 1$. So the $(i, j)$ entry of $E_{pq}E_{pq}^T$ for $i \in \{1, 2, \ldots, n\} \setminus \{p, q\}$ is

$$\sum_{k=1}^{n} e_{ik}^T e_{kj} = e_{ij}^T e_{ij} = 0 \text{ for } i \neq j$$

but $e_{ij} = 1$ if $i = j$. Thus

$$\sum_{k=1}^{n} e_{ik}^T e_{kj} = e_{ij}^T e_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

That is, the $(i, j)$ entry of $E_{pq}E_{pq}^T$ is 0 if $i \neq j$ and 1 is $i = j$; that is, $E_{pq}E_{pq}^T = I_n$ and $E_{pq}$ is orthogonal.

Theorem 3.7.3 (continued 2)

Theorem 3.7.3. Every permutation matrix is orthogonal.

Proof (continued). Second, if $E$ is an elementary permutation matrix resulting from interchanging columns $p$ and $q$ in $I_n$, then $E$ is, similarly, an orthogonal matrix.

So if $P$ is a permutation matrix, that is if $P = E_1 E_2 \cdots E_{\ell}$ for elementary permutation matrices $E_1, E_2, \ldots, E_{\ell}$ (where these correspond to or or column interchanges) then

$$P^T = (E_1 E_2 \cdots E_{\ell})^T = E_{\ell}^T E_{\ell-1}^T \cdots E_1^T$$

$$= E_{\ell}^{-1} E_{\ell-1}^{-1} \cdots E_1^{-1}$$

since each $E_i$ is orthogonal from above

$$= (E_1 E_2 \cdots E_{\ell})^{-1} = P^{-1},$$

and so $PP^T = PP^{-1} = I$. That is, $P$ is orthogonal. \qed