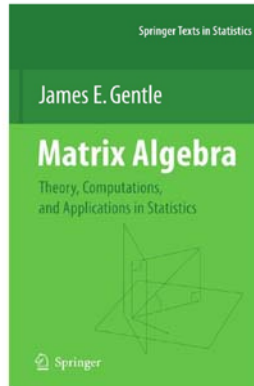


# Theory of Matrices

## Chapter 4. Vector/Matrix Derivatives and Integrals 4.2. Types of Differentiation—Proofs of Theorems



## Theorem 4.2.1

**Theorem 4.2.1.** Differentiation of scalar valued function  $f$  satisfies the following.

- (1)  $\frac{\partial f}{\partial X^T} = \left(\frac{\partial f}{\partial X}\right)^T$ .
- (2) For  $X$  square and  $f(X) = \text{tr}(X)$ ,  $\frac{\partial f}{\partial X} = \mathcal{I}$ .
- (3) For  $AX$  a square matrix where  $A$  is constant,  $\frac{\partial[\text{tr}(AX)]}{\partial X} = A^T$ .
- (4)  $\frac{\partial[\text{tr}(X^T X)]}{\partial X} = 2X$ .
- (5) With  $a$  and  $b$  constant vectors,  $\frac{\partial[a^T X b]}{\partial X} = ab^T$ .
- (6)  $\frac{\partial[\det(X)]}{\partial X} = (\text{adj}(X))^T$ .

## Theorem 4.2.1 (continued 1)

- (1)  $\frac{\partial f}{\partial X^T} = \left(\frac{\partial f}{\partial X}\right)^T$ .
- (2) For  $X$  square and  $f(X) = \text{tr}(X)$ ,  $\frac{\partial f}{\partial X} = \mathcal{I}$ .

**Proof. (1)** From the definition, with  $X^T = [x_{ij}^T]$  where  $x_{ij}^T = x_{ji}$ , we have

$$\frac{\partial f}{\partial X^T} = \left[ \frac{\partial f}{\partial x_{ij}^T} \right] = \left[ \frac{\partial f}{\partial x_{ij}} \right] = \left[ \frac{\partial f}{\partial x_{ij}} \right]^T = \left(\frac{\partial f}{\partial X}\right)^T.$$

**(2)** With  $X$  square and  $f(X) = \text{tr}(X) = \sum_{k=1}^n x_{kk}$  we have

$$\frac{\partial f}{\partial X} = \left[ \frac{\partial[\sum x_{kk}]}{\partial x_{ij}} \right] = \left[ \sum_{k=1}^n \frac{\partial x_{kk}}{\partial x_{ij}} \right] = \mathcal{I}$$

since  $\frac{\partial x_{kk}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (i,j) = (k,k) \\ 0 & \text{if } (i,j) \neq (k,k) \end{cases}$  for  $k = 1, 2, \dots, n$ ; that is,

$$\frac{\partial[\text{tr}(X)]}{\partial X} = \mathcal{I}.$$

## Theorem 4.2.1 (continued 2)

- (3) For  $AX$  a square matrix where  $A$  is constant,  $\frac{\partial[\text{tr}(AX)]}{\partial X} = A^T$ .

**Proof. (3)** For  $AX$  a square matrix where  $A$  is constant, the diagonal entries are  $\sum_{\ell=1}^n a_{k\ell} x_{\ell k}$  for  $k = 1, 2, \dots, n$ . So

$$\frac{\partial[\text{tr}(AX)]}{\partial X} = \left[ \frac{\partial}{\partial x_{ij}} \left[ \sum_{k=1}^n \sum_{\ell=1}^n a_{k\ell} x_{\ell k} \right] \right] = [a_{ji}] = A^T$$

since  $\frac{\partial}{\partial x_{ij}} [\sum_{k=1}^n \sum_{\ell=1}^n a_{k\ell} x_{\ell k}] = \sum_{k=1}^n \sum_{\ell=1}^n a_{k\ell} \frac{\partial x_{\ell k}}{\partial x_{ij}}$  and

$$\frac{\partial x_{\ell k}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (\ell, k) = (i, j) \\ 0 & \text{if } (\ell, k) \neq (i, j) \end{cases}$$

## Theorem 4.2.1 (continued 3)

$$(4) \frac{\partial[\text{tr}(X^T X)]}{\partial X} = 2X.$$

**Proof. (4)** In  $X^T X$ , the diagonal entries are  $\sum_{\ell=1}^n x_{\ell k} x_{\ell k} = \sum_{\ell=1}^n (x_{\ell k})^2$  and so

$$\frac{\partial[\text{tr}(X^T X)]}{\partial X} = \left[ \frac{\partial}{\partial x_{ij}} \left[ \sum_{k=1}^n \sum_{\ell=1}^n x_{\ell k}^2 \right] \right] = [2x_{ij}] = 2X$$

$$\text{since } \frac{\partial x_{\ell k}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (\ell, k) = (i, j) \\ 0 & \text{if } (\ell, k) \neq (i, j) \end{cases}.$$

## Theorem 4.2.1 (continued 4)

$$(5) \text{ With } a \text{ and } b \text{ constant vectors, } \frac{\partial[a^T X b]}{\partial X} = ab^T.$$

**Proof. (5)** With  $a = [a_1, a_2, \dots, a_n]$  and  $b = [b_1, b_2, \dots, b_n]$  constant vectors, the matrix  $a^T X b$  is a  $1 \times 1$  matrix. Now the  $k$ th entry of  $a^T X$  is  $\sum_{\ell=1}^n a_{\ell} x_{\ell k}$  and so

$$a^T X b = \left[ \sum_{k=1}^n \left( \sum_{\ell=1}^n a_{\ell} x_{\ell k} \right) b_k \right].$$

Therefore  $\frac{\partial[a^T X b]}{\partial x_{ij}} = a_i b_j$  since  $\frac{\partial x_{\ell k}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (\ell, k) = (i, j) \\ 0 & \text{if } (\ell, k) \neq (i, j) \end{cases}$ . So

$$\frac{\partial[a^T X b]}{\partial X} = [a_i b_j] = ab^T.$$

## Theorem 4.2.1 (continued 5)

$$(6) \frac{\partial[\det(X)]}{\partial X} = (\text{adj}(X))^T.$$

**Proof. (6)** By Theorem 3.1.F, we can express  $\det(A)$  by expansion along row  $k$  as  $\det(X) = \sum_{k=1}^n x_{kj} \chi_{kj}$  where  $\chi_{kj}$  is the cofactor of  $x_{kj}$ . None of the cofactors  $\chi_{kj}$  involve  $x_{ij}$  so the only occurrence of  $x_{ij}$  in this representation of  $\det(X)$  is when  $k = i$  in the term  $x_{ij} \chi_{ij}$ . Hence,

$\frac{\partial[\det(X)]}{\partial x_{ij}} = \chi_{ij}$  and  $\frac{\partial[\det(X)]}{\partial X} = [\chi_{ij}]$ . Recall that the adjoint of  $X$  is

$\text{adj}(X) = [\chi_{ij}]^T$  and so  $\frac{\partial[\det(X)]}{\partial X} = [\chi_{ij}] = (\text{adj}(X))^T$ .  $\square$