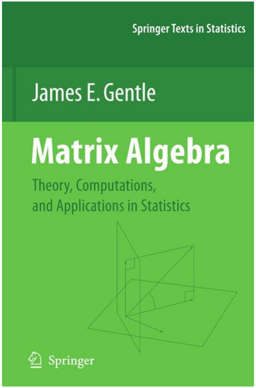
Theory of Matrices

Chapter 4. Vector/Matrix Derivatives and Integrals

4.2. Types of Differentiation—Proofs of Theorems



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Theorem 4.2.1

Theorem 4.2.1. Differentiation of scalar valued function f satisfies the following.

(1) $\dfrac{\partial f}{\partial X^T} = \left(\dfrac{\partial f}{\partial X}\right)^T$.

(2) For X square and $f(X) = \text{tr}(X)$, $\dfrac{\partial f}{\partial X} = \mathcal{I}$.

(3) For AX a square matrix where A is constant, $\dfrac{\partial[\text{tr}(AX)]}{\partial X} = A^T$.

(4) $\dfrac{\partial[\text{tr}(X^T X)]}{\partial X} = 2X$.

(5) With a and b constant vectors, $\dfrac{\partial[a^T X b]}{\partial X} = ab^T$.

(6) $\dfrac{\partial[\det(X)]}{\partial X} = (\text{adj}(X))^T$.

Theorem 4.2.1 (continued 1)

(1) $\dfrac{\partial f}{\partial X^T} = \left(\dfrac{\partial f}{\partial X}\right)^T$.

(2) For X square and $f(X) = \text{tr}(X)$, $\dfrac{\partial f}{\partial X} = \mathcal{I}$.

Proof. **(1)** From the definition, with $X^T = [x_{ij}^T]$ where $x_{ij}^T = x_{ji}$, we have

$$\frac{\partial f}{\partial X^T} = \left[\frac{\partial f}{\partial x_{ij}^T}\right] = \left[\frac{\partial f}{\partial x_{ij}}\right] = \left[\frac{\partial f}{\partial x_{ij}}\right]^T = \left(\frac{\partial f}{\partial X}\right)^T.$$

(2) With X square and $f(X) = \text{tr}(X) = \sum_{k=1}^n x_{kk}$ we have

$$\frac{\partial f}{\partial X} = \left[\frac{\partial[\sum x_{kk}]}{\partial x_{ij}}\right] = \left[\sum_{k=1}^n \frac{\partial x_{kk}}{\partial x_{ij}}\right] = \mathcal{I}$$

since $\frac{\partial x_{kk}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (i,j) = (k,k) \\ 0 & \text{if } (i,j) \neq (k,k) \end{cases}$ for $k = 1, 2, \ldots, n$; that is, $\frac{\partial[\text{tr}(X)]}{\partial X} = \mathcal{I}$.

Theorem 4.2.1 (continued 1)

(1) $\dfrac{\partial f}{\partial X^T} = \left(\dfrac{\partial f}{\partial X}\right)^T$.

(2) For X square and $f(X) = \text{tr}(X)$, $\dfrac{\partial f}{\partial X} = \mathcal{I}$.

Proof. (1) From the definition, with $X^T = [x_{ij}^T]$ where $x_{ij}^T = x_{ji}$, we have

$$\frac{\partial f}{\partial X^T} = \left[\frac{\partial f}{\partial x_{ij}^T}\right] = \left[\frac{\partial f}{\partial x_{ij}}\right] = \left[\frac{\partial f}{\partial x_{ij}}\right]^T = \left(\frac{\partial f}{\partial X}\right)^T.$$

(2) With X square and $f(X) = \text{tr}(X) = \sum_{k=1}^{n} x_{kk}$ we have

$$\frac{\partial f}{\partial X} = \left[\frac{\partial[\sum x_{kk}]}{\partial x_{ij}}\right] = \left[\sum_{k=1}^{n} \frac{\partial x_{kk}}{\partial x_{ij}}\right] = \mathcal{I}$$

since $\frac{\partial x_{kk}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (i,j) = (k,k) \\ 0 & \text{if } (i,j) \neq (k,k) \end{cases}$ for $k = 1, 2, \ldots, n$; that is, $\frac{\partial[\text{tr}(X)]}{\partial X} = \mathcal{I}$.

Theorem 4.2.1 (continued 2)

(3) For AX a square matrix where A is constant,
$$\frac{\partial[\mathrm{tr}(AX)]}{\partial X} = A^T.$$

Proof. (3) For AX a square matrix where A is constant, the diagonal entries are $\sum_{\ell=1}^{n} a_{k\ell}x_{\ell k}$ for $k = 1, 2, \ldots, n$. So

$$\frac{\partial[\mathrm{tr}(AX)]}{\partial X} = \left[\frac{\partial}{\partial x_{ij}}\left[\sum_{k=1}^{n}\sum_{\ell=1}^{n} a_{k\ell}x_{\ell k}\right]\right] = [a_{ji}] = A^T$$

since $\frac{\partial}{\partial x_{ij}}\left[\sum_{k=1}^{n}\sum_{\ell=1}^{n} a_{k\ell}x_{\ell k}\right] = \sum_{k=1}^{n}\sum_{\ell=1}^{n} a_{k\ell}\frac{\partial x_{\ell k}}{\partial x_{ij}}$ and
$\frac{\partial x_{\ell k}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (\ell, k) = (i, j) \\ 0 & \text{if } (\ell, k) \neq (i, j) \end{cases}.$

Theorem 4.2.1 (continued 3)

(4) $$\frac{\partial[\text{tr}(X^T X)]}{\partial X} = 2X.$$

Proof. (4) In $X^T X$, the diagonal entries are $\sum_{\ell=1}^{n} x_{\ell k} x_{\ell k} = \sum_{\ell=1}^{n} (x_{\ell k})^2$ and so

$$\frac{\partial[\text{tr}(X^T X)]}{\partial X} = \left[\frac{\partial}{\partial x_{ij}}\left[\sum_{k=1}^{n}\sum_{\ell=1}^{n} x_{\ell k})^2\right]\right] = [2x_{ij}] = 2X$$

since $\frac{\partial x_{\ell k}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (\ell, k) = (i, j) \\ 0 & \text{if } (\ell, k) \neq (i, j) \end{cases}$.

Theorem 4.2.1 (continued 4)

(5) With a and b constant vectors, $\frac{\partial[a^T Xb]}{\partial X} = ab^T$.

Proof. (5) With $a = [a_1, a_2, \ldots, a_n]$ and $b = [b_1, b_2, \ldots b_n]$ constant vectors, the matrix $a^T Xb$ is a 1×1 matrix. Now the kth entry of $a^T X$ is $\sum_{\ell=1}^{n} a_\ell x_{\ell k}$ and so

$$a^T Xb = \left[\sum_{k=1}^{n} \left(\sum_{\ell=1}^{n} a_\ell x_{\ell k}\right) b_k\right].$$

Therefore $\frac{\partial[a^T Xb]}{\partial x_{ij}} = a_i b_j$ since $\frac{\partial x_{\ell k}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (\ell, k) = (i, j) \\ 0 & \text{if } (\ell, k) \neq (i, j) \end{cases}$. So $\frac{\partial[a^T Xb]}{\partial X} = [a_i b_j] = ab^T$.

Theorem 4.2.1 (continued 4)

(5) With a and b constant vectors, $\dfrac{\partial[a^T X b]}{\partial X} = ab^T$.

Proof. (5) With $a = [a_1, a_2, \ldots, a_n]$ and $b = [b_1, b_2, \ldots b_n]$ constant vectors, the matrix $a^T X b$ is a 1×1 matrix. Now the kth entry of $a^T X$ is $\sum_{\ell=1}^n a_\ell x_{\ell k}$ and so

$$a^T X b = \left[\sum_{k=1}^n \left(\sum_{\ell=1}^n a_\ell x_{\ell k} \right) b_k \right].$$

Therefore $\frac{\partial[a^T X b]}{\partial x_{ij}} = a_i b_j$ since $\frac{\partial x_{\ell k}}{\partial x_{ij}} = \begin{cases} 1 & \text{if } (\ell, k) = (i, j) \\ 0 & \text{if } (\ell, k) \neq (i, j) \end{cases}$. So $\frac{\partial[a^T X b]}{\partial X} = [a_i b_j] = ab^T$.

Theorem 4.2.1 (continued 5)

$$(6)\quad \frac{\partial[\det(X)]}{\partial X} = (\mathrm{adj}(X))^T.$$

Proof. (6) By Theorem 3.1.F, we can express $\det(A)$ by expansion along row k as $\det(X) = \sum_{k=1}^{n} x_{kj}\chi_{kj}$ where χ_{kj} is the cofactor of x_{kj}. None of the cofactors χ_{kj} involve x_{ij} so the only occurrence of x_{ij} in this representation of $\det(X)$ is when $k = i$ in the term $x_{ij}\chi_{ij}$. Hence, $\frac{\partial[\det(X)]}{\partial x_{ij}} = \chi_{ij}$ and $\frac{\partial[\det(X)]}{\partial X} = [\chi_{ij}]$. Recall that the adjoint of X is $\mathrm{adj}(X) = [\chi_{ij}]^T$ and so $\frac{\partial[\det(X)]}{\partial X} = [\chi_{ij}] = (\mathrm{adj}(X))^T$. □

Theorem 4.2.1 (continued 5)

(6) $\dfrac{\partial[\det(X)]}{\partial X} = (\mathrm{adj}(X))^T.$

Proof. (6) By Theorem 3.1.F, we can express $\det(A)$ by expansion along row k as $\det(X) = \sum_{k=1}^{n} x_{kj}\chi_{kj}$ where χ_{kj} is the cofactor of x_{kj}. None of the cofactors χ_{kj} involve x_{ij} so the only occurrence of x_{ij} in this representation of $\det(X)$ is when $k = i$ in the term $x_{ij}\chi_{ij}$. Hence, $\frac{\partial[\det(X)]}{\partial x_{ij}} = \chi_{ij}$ and $\frac{\partial[\det(X)]}{\partial X} = [\chi_{ij}]$. Recall that the adjoint of X is $\mathrm{adj}(X) = [\chi_{ij}]^T$ and so $\frac{\partial[\det(X)]}{\partial X} = [\chi_{ij}] = (\mathrm{adj}(X))^T$. □