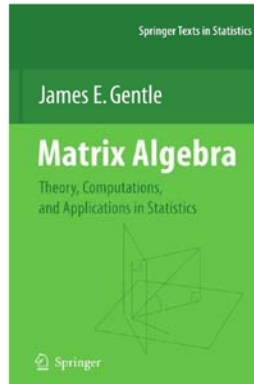


Theory of Matrices

Chapter 4. Vector/Matrix Derivatives and Integrals 4.5. Integration and Expectation—Proofs of Theorems



Theorem 4.5.1

Theorem 4.5.1. For $A(x) = [a_{ij}(x)]$ an $n \times n$ matrix function of scalar variable x , we have

$$\int_a^b \text{tr}(A(x)) \, dx = \text{tr} \left(\int_a^b A(x) \, dx \right).$$

Proof. Recall that $\text{tr}(A) = \text{tr}([a_{ij}]) = \sum_{i=1}^n a_{ii}$. So

$$\begin{aligned} \int_a^b \text{tr}(A(x)) \, dx &= \int_a^b \left(\sum_{i=1}^n a_{ii}(x) \right) \, dx \\ &= \sum_{i=1}^n \left(\int_a^b a_{ii}(x) \, dx \right) \text{ by the linearity properties} \\ &\quad \text{of definite integrals} \\ &= \text{tr} \left(\left[\int_a^b a_{ij}(x) \, dx \right] \right) = \text{tr} \left(\int_a^b A(x) \, dx \right). \end{aligned}$$

□

Theorem 4.5.3

Theorem 4.5.3. Atiken's Integral.

For Σ^{-1} a symmetric positive definite $d \times d$ matrix, is a constant d -vector, and $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} e^{-(x-\mu)^T \Sigma^{-1} (x-\mu)/2} \, dx = (2\pi)^{d/2} (\det(\Sigma))^{1/2}.$$

Proof. We know that $\int_{\mathbb{R}} e^{-x^2} \, dx = \sqrt{2\pi}$, based on the standard normal distribution. If we let $y = x - \mu$ then we also have

$$\int_{\mathbb{R}} e^{-y^2} \, dy = \int_{\mathbb{R}} e^{-(x-\mu)^2} \, dx = \sqrt{2\pi}.$$

Now we make a change of variables and let $y = x - \mu$ where $y \in \mathbb{R}^d$. Then as x ranges over all of \mathbb{R}^d , so does y and hence

$$\int_{\mathbb{R}^d} e^{-(x-\mu)^T \Sigma^{-1} (x-\mu)/2} \, dx = \int_{\mathbb{R}^d} e^{-y^T \Sigma^{-1} y/2} \, dy.$$

Theorem 4.5.3 (continued 1)

Theorem 4.5.3. Atiken's Integral.

For Σ^{-1} a symmetric positive definite $d \times d$ matrix, is a constant d -vector, and $x \in \mathbb{R}^d$ we have

Proof (continued). By Theorem 3.8.15(1), since Σ^{-1} is symmetric and positive definite, then $P^T \Sigma^{-1} P = \mathcal{I}$ for some nonsingular P . We now make another change of variables to $z = P^{-1}y$ (and so $y = Pz$). The change of variables requires that we introduce the Jacobian, as described at the beginning of the section (but now in d -dimensions) we need $\det(P)$. Since $P^T \Sigma^{-1} P = \mathcal{I}$ then by Theorem 3.2.4, $\det(P^T) \det(\Sigma^{-1}) \det(P) = \det(\mathcal{I}) = 1$, and since $\det(P^T) = \det(P)$ by Theorem 3.1.A and $\det(\Sigma^{-1}) = 1/\det(\Sigma)$ (also by Theorem 3.1.A), then $(\det(P))^2 = \det(\Sigma)$ or $\det(P) = (\det(\Sigma))^{1/2}$.

Theorem 4.5.3 (continued 2)

Proof (continued). So

$$\begin{aligned}
 \int_{\mathbb{R}^d} e^{-y^T \Sigma^{-1} y / 2} dt &= \int_{\mathbb{R}^d} e^{-(Pz)^T \Sigma^{-1} (Pz) / 2} (\det(\Sigma))^{1/2} dz \\
 &= \int_{\mathbb{R}^d} e^{-z^T (P^T \Sigma^{-1} P) z / 2} (\det(\Sigma))^{1/2} dz \\
 &= \int_{\mathbb{R}^d} e^{z^T z / 2} (\det(\Sigma))^{1/2} dz \text{ since } P^T \Sigma^{-1} P = \mathcal{I} \\
 &= (\det(\Sigma))^{1/2} \int_{\mathbb{R}^d} e^{-|z|^2 / 2} dz \\
 &= (\det(\Sigma))^{1/2} \int_{\mathbb{R}^d} e^{-(z_1^2 + z_2^2 + \dots + z_d^2) / 2} dz \\
 &\quad \text{where vector } z \text{ has components } z_i
 \end{aligned}$$

...

Theorem 4.5.3 (continued 3)

Theorem 4.5.3. Atiken's Integral.

For Σ^{-1} a symmetric positive definite $d \times d$ matrix, is a constant d -vector, and $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} e^{-(x-\mu)^T \Sigma^{-1} (x-\mu) / 2} dx = (2\pi)^{d/2} (\det(\Sigma))^{1/2}.$$

Proof (continued). ...

$$\begin{aligned}
 \int_{\mathbb{R}^d} e^{-(x-\mu)^T \Sigma^{-1} (x-\mu) / 2} dx &= \int_{\mathbb{R}^d} e^{-y^T \Sigma^{-1} y / 2} dy \\
 &= (\det(\Sigma))^{1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} e^{-z_1^2} e^{-z_2^2} \dots e^{-z_d^2} dz_1 dz_2 \dots dz_d \\
 &= (\det(\Sigma))^{1/2} \int_{\mathbb{R}} e^{-z_1^2} dz_1 \int_{\mathbb{R}} e^{-z_2^2} dz_2 \dots \int_{\mathbb{R}} e^{-z_d^2} dz_d \\
 &= (\det(\Sigma))^{1/2} ((2\pi)^{1/2})^d \text{ as described above} \\
 &= (2\pi)^{d/2} (\det(\Sigma))^{1/2}. \quad \square
 \end{aligned}$$