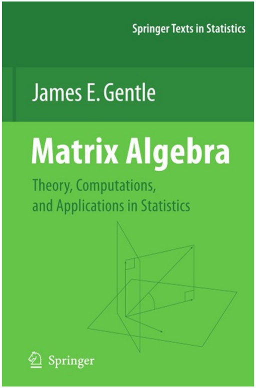
Theory of Matrices

Chapter 4. Vector/Matrix Derivatives and Integrals

4.5. Integration and Expectation—Proofs of Theorems



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Theorem 4.5.1

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$$\int_a^b \mathrm{tr}(A(x))\,dx = \mathrm{tr}\left(\int_a^b A(x)\,dx\right).$$

Proof. Recall that $\mathrm{tr}(A) = \mathrm{tr}([a_{ij}]) = \sum_{i=1}^n a_{ii}$. So

$$\begin{aligned}
\int_a^b \mathrm{tr}(A(x))\,dx &= \int_a^b \left(\sum_{i=1}^n a_{ii}(x)\right) dx \\
&= \sum_{i=1}^n \left(\int_a^b a_{ii}(x)\,dx\right) \text{ by the linearity properties} \\
&\quad \text{of definite integrals} \\
&= \mathrm{tr}\left(\left[\int_a^b a_{ij}(x)\right]\right) = \mathrm{tr}\left(\int_a^b A(x)\,dx\right).
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Theorem 4.5.3

Theorem 4.5.3. Atiken's Integral.
For Σ^{-1} a symmetric positive definite $d \times d$ matrix, is a constant d-vector, and $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} e^{-(x-\mu)^T \Sigma^{-1}(x-\mu)/2}\, dx = (2\pi)^{d/2}(\det(\Sigma))^{1/2}.$$

Proof. We know that $\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{2\pi}$, based on the standard normal distribution. If we let $y = x - \mu$ then we also have

$$\int_{\mathbb{R}} e^{-y^2}\, dy = \int_{\mathbb{R}} e^{-(x-\mu)^2}\, dx = \sqrt{2\pi}.$$

Now we make a change of variables and let $y = x - \mu$ where $y \in \mathbb{R}^d$. Then as x ranges over all of $\mathbb{R}^d$, so does y and hence

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Proof (continued). By Theorem 3.8.15(1), since Σ^{-1} is symmetric and positive definite, then $P^T \Sigma^{-1} P = \mathcal{I}$ for some nonsingular P. We now make another change of variables to $z = P^{-1} y$ (and so $y = Pz$). The change of variables requires that we introduce the Jacobian, as described at the beginning of the section (but now in d-dimensions) we need $\det(P)$. Since $P^T \Sigma^{-1} P = \mathcal{I}$ then by Theorem 3.2.4, $\det(P^T)\det(\Sigma^{-1})\det(P) = \det(\mathcal{I}) = 1$, and since $\det(P^T) = \det(P)$ by Theorem 3.1.A and $\det(\Sigma^{-1}) = a/\det(\Sigma)$ (also by Theorem 3.1.A), then $(\det(P))^2 = \det(\Sigma)$ or $\det(P) = (\det(\Sigma))^{1/2}$.

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Theorem 4.5.3 (continued 2)

Proof (continued). So

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-y^T\Sigma^{-1}y/2}\,dt &= \int_{\mathbb{R}^d} e^{-(Pz)^T\Sigma^{-1}(Pz)/2}(\det(\Sigma))^{1/2}\,dz \\
&= \int_{\mathbb{R}^d} e^{-z^T(P^T\Sigma^{-1}P)z/2}(\det(\Sigma))^{1/2}\,dz \\
&= \int_{\mathbb{R}^d} e^{z^Tz/2}(\det(\Sigma))^{1/2}\,dz \text{ since } P^T\Sigma^{-1}P = \mathcal{I} \\
&= (\det(\Sigma))^{1/2}\int_{\mathbb{R}^d} e^{-|z|^2/2}\,dz \\
&= (\det(\Sigma))^{1/2}\int_{\mathbb{R}^d} e^{-(z_1^2+z_2^2+\cdots+z_d^2)/2}\,dz \\
&\qquad \text{where vector } z \text{ has components } z_i
\end{aligned}$$

...

Theorem 4.5.3 (continued 3)

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For Σ^{-1} a symmetric positive definite $d \times d$ matrix, is a constant d-vector, and $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} e^{-(x-\mu)^T \Sigma^{-1}(x-\mu)/2}\, dx = (2\pi)^{d/2}(\det(\Sigma))^{1/2}.$$

Proof (continued). ...

$$\int_{\mathbb{R}^d} e^{-(x-\mu)^T \Sigma^{-1}(x-\mu)/2}\, dx = \int_{\mathbb{R}^d} e^{-y^T \Sigma^{-1} y/2}\, dy$$

$$\begin{aligned}
&= (\det(\Sigma))^{1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-z_1^2} e^{-z_2^2} \cdots e^{-z_d^2}\, dz_1\, dz_2 \cdots dz_d \\
&= (\det(\Sigma))^{1/2} \int_{\mathbb{R}} e^{-z_1^2}\, dz_1 \int_{\mathbb{R}} e^{-z_2^2}\, dz_2 \cdots \int_{\mathbb{R}} e^{-z_d^2}\, dz_d \\
&= (\det(\Sigma))^{1/2}((2\pi)^{1/2})^d \text{ as described above} \\
&= (2\pi)^{d/2}(\det(\Sigma))^{1/2}. \quad \square
\end{aligned}$$