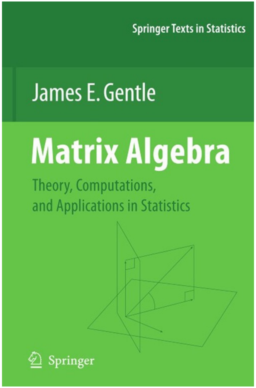
Theory of Matrices

Chapter 5. Matrix Transformations and Factorizations

5.6. LU and LDU Factorizations—Proofs of Theorems



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Proof. As described in the previous note, there is a sequence of $n \times n$ elementary matrices E_i such that $E_h E_{h-1} \cdots E_2 E_1 A = U$ where each E_i is an elementary matrix associated with the elementary row operation of row addition. Since U is upper triangular then the row operations need only involve adding a multiple of one row to a lower row ($R_p \to R_p + sR_q$ where $p > q$).

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Proof (continued). Following the Gauss-Jordan method (where the first column is processed from top to bottom, then the second column, etc.), then matrix E_h^{-1} differs from the identity only in row n and column $n-1$ (though it is also possible that this entry is 0). Then E_{h-1}^{-1} differs from the identity only in row n and column $n-2$, and E_{h-2}^{-1} differs from the identity only in row $n-1$ and column $n-2$, and so forth. So as the inverse matrices are multiplied together in product $E_1^{-1}E_2^{-1}\cdots E_{h-1}^{-1}E_h^{-1}$ the entries in the product are filled in as follows.

Theorem 5.6.A (continued 1)

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Theorem 5.6.A (continued 2)

Proof (continued). The E_k^{-1}'s are applied in the order $E_h^{-1}, E_{h-1}^{-1}, E_{h-2}^{-1}, \ldots, E_1^{-1}$ and this order (and the entries they affect) is given by the colored numbers; the colored numbers are not values!

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 10 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 9 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 8 & 5 & 3 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 7 & 4 & 2 & 1 & 1 \end{bmatrix}$$

Therefore $L = E_1^{-1} E_2^{-1} \cdots E_{h-1}^{-1} E_h^{-1}$ is lower triangular and $A = LU$. $\square$

Theorem 5.6.B

Theorem 5.6.B. Unique Factorization.
Let A be an $n \times m$ matrix. When a factorization $A = LDU$ exists where

1. L is a lower triangular $n \times n$ matrix with all main diagonal entries 1,
2. U is upper triangular $n \times m$ matrix with all diagonal entries 1, and
3. D is a diagonal $n \times n$ matrix with all main diagonal entries nonzero,

it is unique.

Proof. Suppose that $A = L_1 D_1 U_1 = L_2 D_2 U_2$ are two such factorizations. Then L_1^{-1} and L_2^{-1} are also lower triangular, D_1^{-1} and D_2^{-1} are both diagonal and U_1^{-1} and U_2^{-1} are both upper triangular. Since the diagonal entries of L_1, L_2, U_1, U_2 are all 1 then the diagonal entries of $L_1^{-1}, L_2^{-1}, U_1^{-1}, U_2^{-1}$ are also all 1.

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Theorem 5.6.B (continued)

Theorem 5.6.B. Unique Factorization.
Let A be a square matrix. When a factorization $A = LDU$ exists where

1. L is a lower triangular matrix with all main diagonal entries 1,
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Proof (continued). Since $A = L_1 D_1 U_1 = L_2 D_2 U_2$, we have $L_2^{-1} L_1 = D_2 U_2 U_1^{-1} D_1^{-1}$. A product of upper/lower triangular matrices is upper/lower triangular, so $L_2^{-1} L_1$ is lower triangular and $D_2 U_2 U_1^{-1} D_1^{-1}$ is upper triangular. Since $L_2^{-1} L_1 = D_2 U_2 U_1^{-1} D_1^{-1}$ then both sides of this equation must be the identity. So $L_2^{-1} L_1 = I$ and $L_1 = L_2$. Similarly, we can conclude $U_1 U_2^{-1} = D_1^{-1} L_1^{-1} L_2 D_2$ and both sides must be the identity. So $U_1 = U_2$.

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Proof (continued). Since $A = L_1 D_1 U_1 = L_2 D_2 U_2$, we have $L_2^{-1} L_1 = D_2 U_2 U_1^{-1} D_1^{-1}$. A product of upper/lower triangular matrices is upper/lower triangular, so $L_2^{-1} L_1$ is lower triangular and $D_2 U_2 U_1^{-1} D_1^{-1}$ is upper triangular. Since $L_2^{-1} L_1 = D_2 U_2 U_1^{-1} D_1^{-1}$ then both sides of this equation must be the identity. So $L_2^{-1} L_1 = I$ and $L_1 = L_2$. Similarly, we can conclude $U_1 U_2^{-1} = D_1^{-1} L_1^{-1} L_2 D_2$ and both sides must be the identity. So $U_1 = U_2$. We then have $L_1 D_1 U_1 = L_1 D_2 U_1$ and since all matrices are invertible, we conclude $D_1 = D_2$. We therefore have $L_1 = L_2$, $U_1 = U_2$, and $D_1 = D_2$. So the factorization of A is unique. $\square$

Theorem 5.6.B (continued)

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Let A be a square matrix. When a factorization $A = LDU$ exists where

1. L is a lower triangular matrix with all main diagonal entries 1,
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Proof (continued). Since $A = L_1 D_1 U_1 = L_2 D_2 U_2$, we have $L_2^{-1} L_1 = D_2 U_2 U_1^{-1} D_1^{-1}$. A product of upper/lower triangular matrices is upper/lower triangular, so $L_2^{-1} L_1$ is lower triangular and $D_2 U_2 U_1^{-1} D_1^{-1}$ is upper triangular. Since $L_2^{-1} L_1 = D_2 U_2 U_1^{-1} D_1^{-1}$ then both sides of this equation must be the identity. So $L_2^{-1} L_1 = I$ and $L_1 = L_2$. Similarly, we can conclude $U_1 U_2^{-1} = D_1^{-1} L_1^{-1} L_2 D_2$ and both sides must be the identity. So $U_1 = U_2$. We then have $L_1 D_1 U_1 = L_1 D_2 U_1$ and since all matrices are invertible, we conclude $D_1 = D_2$. We therefore have $L_1 = L_2$, $U_1 = U_2$, and $D_1 = D_2$. So the factorization of A is unique. $\square$