Section 3.10. Approximation of Matrices

Note. When we say "approximate a matrix" we mean, for given matrix A, to find a matrix $\tilde{A}$ such that $\|A - \tilde{A}\|$ is minimal for all matrices $\tilde{A}$ satisfying some given property. In this brief section (which is just over 2 pages long) we give a best approximation with respect to the proerty of rank. The norm we use is the Frobenius norm.

Theorem 3.10.1. Let A be an $n \times m$ matrix of rank r with singular value decomposition (which exists by Theorem 3.8.16)

$$A = U \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} V^T$$

where $D_r = \mathrm{diag}(d_1, d_2, \ldots, d_r)$ and the singular values are indexed so that $d_1 \geq d_2 \geq \cdots \geq d_r \geq 0$. Then for all $n \times m$ matrices X with rank $k < r$ we have $\|A - X\|_F^2 \geq \sum_{i=k+1}^{r} d_i^2$ and this inequality reduces to equality (giving a best approximation) for $X = \tilde{A}$ where

$$A = U \begin{bmatrix} D_k & 0 \\ 0 & 0 \end{bmatrix} V^T$$

and $D_k = \mathrm{diag}(d_1, d_2, \ldots, d_k)$.

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