Section 3.6. Generalized Inverses

Note. For A invertible, the unique solution to the system $Ax = b$ is $x = A^{-1}b$. We saw in Theorem 3.5.3(1) that if A^- is a generalized inverse of A then a solution to $Ax = b$ is $x = A^-b$. In this section we explore the generalized inverse of a matrix and show that such a matrix always exists. We introduce the pseudoinverse (or Moore-Penrose inverse) of a matrix, show that it exists and is unique for a given matrix.

Note. Gentle claims on page 101 that the results of Theorem 3.3.17 for the inverse of a matrix also hold for the generalized inverse of a matrix. However, in the errata to the text (see online errata) this statement is corrected to read that the properties of Theorem 3.3.17 *do not* in general hold for the generalized inverse. Gentle also gives a formula for a generalized inverse of a partitioned matrix on page 101 (see equation (3.165) and Exercise 3.14). I question the accuracy of this formula (and it claims the existence of A^- in terms of generalized inverses of parts of a partition of A, so does not actually establish the existence of generalized inverses). We now establish the existence of a generalized inverse for any matrix A and we'll see that A^- is not unique.

Note. Recall that for $n \times m$ matrix A, a $m \times n$ matrix A^- is a generalized inverse of A if $AA^-A = A$. Notice that if $A = 0$ (and so $\text{rank}(A) = 0$) then every $m \times n$ matrix is a generalized inverse of A!

Note. If (as correctly pointed out by Gentle on page 101) for $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11} is of full rank and the same rank as A, then a generalized inverse of A is $A^- = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ since

$$AA^-A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A$$

because for A_{11} full rank we have $A_{22} = A_{21}A_{11}^{-1}A_{12}$ (see the first note on page 2 of the class notes for Section 3.4). We can similarly establish the existence of a generalized inverse for any $n \times m$ matrix A as follows.

Theorem 3.6.1. Let A be an $n \times m$ matrix. Then a generalized inverse of A exists.

Note. We see from the proof of Theorem 3.6.1 that there is a unique generalized inverse of $n \times m$ matrix A if and only if $A = PI_rQ$ where P and Q are products of elementary matrices and hence $\text{rank}(A) = r$ and A is equivalent to I_r. That is, the generalized inverse of A is unique if and only if A is invertible. We now explore a different kind of inverse which is unique.

Definition. For $n \times m$ matrix A, a *pseudoinverse* of A (or *Moore-Penrose inverse* of A), denoted A^+, is a $m \times n$ matrix satisfying:

(1) $AA^+A = A$,

(2) $A^+AA^+ = A^+$,

(3) A^+A is symmetric, and

(4) AA^+ is symmetric.

Note. If $A = 0$, then $A^+ = 0$. Notice that from condition (2), $A^+AA^+ = A^+$, we see that A^+ must be 0 when $A = 0$. We now address the pseudoinverse of A for $A \neq 0$ (i.e., for $\text{rank}(A) > 0$).

Theorem 3.6.2. Every matrix A with $\text{rank}(A) = r > 0$ has a pseudoinverse given be $A^+ = R^T(L^TAR^T)^{-1}L^T$ where $A = LR$ is a full rank factorization of A (such a factorization exists as shown in equations $(**)$ and $(***)$ of Section 3.4).

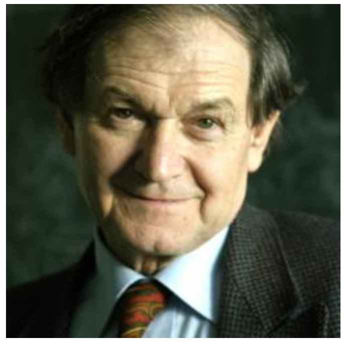
Theorem 3.6.3. For any matrix A, the pseudoinverse A^+ is unique.

Note. For invertible A, A^{-1} is a pseudoinverse and since A^+ is unique by Theorem 3.6.3, for invertible A we have $A^+ = A^{-1}$.

Note. If A^+ is the pseudoinverse of A then $x = A^+b$ is a solution for $Ax = b$ since by Property (1), $AA^+A = A$ so that for system $Ax = b$ we have $(AA^+A)x = b$ or $AA^+(Ax) = b$ or $AA^+b = b$ or $A(A^+b) = b$.

Note. The four properties in the definition of pseudoinverse are called the *Penrose equations*. Roger Penrose in 1955 showed that every finite matrix (with real or complex entries; in the complex case, symmetry is replaced with conjugate symmetry) has a unique pseudoinverse satisfying the four equations (R. Penrose, "A Generalized Inverse for Matrices," *Proceedings of the Cambridge Philosophical Society* **51** (1955), 406–413). This idea was addressed in 1920 by E. H. Moore ("On the Reciprocal of the General Algebraic Matrix," *Bulletin of the American Mathematical Society* **26** (1920), 394–395) and this is why the pseudoinverse is often called the *Moore-Penrose inverse*. C. C. MacDuffee apparently was the first to give the formula in Theorem 3.6.2 in 1959 in a private communication. The information of this note is based on Chapter 1 "Existence and Construction of Generalized Inverses" in A. Ben-Israel and T. Greville's *Generalized Inverses: Theory and Applications*, 2nd Edition, Springer (2003).

Note. Roger Penrose, born in 1931, is currently (summer 2020) an emeritus member of the Mathematical Institute at the University of Oxford. He did his Ph.D. work at the University of Cambridge in the area of algebraic geometry. He is probably best known for his work in physics; with Stephen Hawking in 1969 he showed that all the matter within a black hole collapses to a singularity of infinite density and zero volume. In addition he developed a method of mapping the regions of space-time surrounding a black hole. This information is from Encyclopedia Britannica's online Roger Penrose biography (accessed 11/30/2017).



This image is from Roger Penrose's faculty website (accessed 6/14/2020).

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