Section 3.7. Orthogonality

Note. We define orthogonal matrices and explore some of their properties. We introduce the orthogonal group $\mathcal{O}(n)$.

Definition. A $n \times m$ matrix with real entries whose rows or columns constitute a set of orthonormal vectors is an *orthogonal matrix* (or *unitary matrix*).

Theorem 3.7.1. Let Q be an $n \times m$ matrix. For $n \leq m$, Q is orthogonal if and only if $QQ^T = I_n$. For $n \geq m$, Q is orthogonal if and only if $Q^TQ = I_m$. A square matrix Q is orthogonal if and only if $QQ^T = Q^TQ = I$ (so a square matrix Q is orthogonal if and only if it is invertible and $Q^{-1} = Q^T$).

Note. It is common to define an orthogonal real matrix as a square matrix A satisfying $A^TA = I$, as given in Theorem 3.7.1. See Section 6.3 "Orthogonal Matrices" in Fraleigh and Beauregard's *Linear Algebra*, 3rd Edition, Addison-Wesley (1995).

Definition. If $n \times m$ matrix Q has complex entries then it is *orthogonal* (or *unitary*) if $QQ^H = I_n$ for $n \leq m$, or $Q^HQ = I_m$ for $n \geq m$, where Q^H is the conjugate transpose of Q (or the "Hermitian" of Q).

Corollary 3.7.2. For Q a square orthogonal matrix, we have $\det(Q) = \pm 1$. For Q an $n \times m$ orthogonal matrix Q with $n \geq m$, we have $\langle Q, Q \rangle = m$.

Theorem 3.7.3. Every permutation matrix is orthogonal.

Theorem 3.7.4. If A and B are orthogonal then the Kronecker product $A \otimes B$ is orthogonal.

Note. The proof of Theorem 3.7.4 is to be given in Exercise 3.7.A. In Exercise 3.7.B, it is to be shown that the set of all $n \times n$ orthogonal matrices form a group.

Definition. The group of all $n \times n$ orthogonal matrices is the *orthogonal group*, denoted $\mathcal{O}(n)$.

Note. We also address the orthogonal group in a supplement to Section 22 of Munkres *Topology* (2nd Edition, Pearson (2000)). See my online notes on 22. Topological Groups.

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